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On Recursive Operations Over Logic LTS †

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Recently, in order to mix algebraic and logic styles of specification in a uniform framework, the notion of a logic labelled transition system (Logic LTS or LLTS for short) has been introduced and explored. A variety of constructors over LLTS, including usual process-algebraic operators, logic connectives (*conjunction* and *disjunction*) and standard temporal operators (*always* and *unless*), have been given. However, no attempt has made so far to develop general theory concerning (nested) recursive operations over LLTS and a few fundamental problems are still open. This paper intends to study this issue in pure process-algebraic style. A few fundamental properties, including precongruence and the uniqueness of consistent solutions for equations, will be established.

1. Introduction

Algebra and logic are two dominant approaches for the specification, verification and systematic development of reactive and concurrent systems. They take different standpoint for looking at specifications and verifications, and offer complementary advantages.

Logical approaches devote themselves to specifying and verifying abstract properties of systems. In such framework, the most common reasonable property of concurrent systems, such as safety, liveness, etc., can be formulated in terms of logic formulas without resorting to operational details and verification is a deductive or model-checking activity. However, due to their global perspective and abstract nature, logical approaches often give little support for modular designing and compositional reasoning.

Algebraic approaches put attention to behavioral aspects of systems, which have tended to use formalisms in algebraic style. These formalisms are referred to as process algebra or process calculus. In such paradigm, a specification and its implementation usually are formulated by terms (expressions) of a formal language built from a number of operators, and the underlying semantics are often assigned operationally. The verification amounts

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to compare terms, which is often referred to as *implementation verification* or *equivalence checking*. Algebraic approaches often support compositional constructing and reasoning, which bring us advantages in developing systems, such as, supporting modular design and verification, avoiding verifying the whole system from scratch when its parts are modified, allowing reusability of proofs and so on (Andersen *et al.* 1994). Thus such approaches offer significant support for rigorous systematic development of reactive and concurrent systems. However, since algebraic approaches specify a system by means of prescribing in detail how the system should behave, it is often difficult for them to describe abstract properties of systems, which is a major disadvantage of such approaches.

In order to take advantage of these two approaches when designing systems, so-called heterogeneous specifications have been proposed, which uniformly integrate these two specification styles. Amongst, based on Büchi automata and labelled transition system (LTS) augmented with a predicate, Cleaveland and Lüttgen provide a semantic framework for heterogenous system design (Cleaveland and Lüttgen 2000, 2002). In this framework, not only usual operational operators but also logic connectives are considered, and must-testing preorder presented in (Nicola and Hennessy 1983) is adopted to capture refinement relation. Unfortunately, this setting does not support compositional reasoning since must-testing preorder is not a precongruence in such situation. Moreover, the logic connective conjunction in this framework lacks the desired property that r is an implementation of a given specification $p \wedge q$ if and only if r implements both p and q.

Recently, Lüttgen and Vogler introduce the notion of a Logic LTS (LLTS), which combines operational and logic styles of specification in one unified framework (Lüttgen and Vogler 2007, 2010, 2011). In addition to usual operational constructors, e.g., CSP-style parallel composition, hiding and so on, logic connectives (conjunction and disjunction) and standard modal operators (*always* and *unless*) are also integrated into this framework. Moreover, the drawbacks in (Cleaveland and Lüttgen 2000, 2002) above mentioned have been remedied by adopting ready-tree semantics (Lüttgen and Vogler 2007). In order to support compositional reasoning in the presence of the parallel constructor, a variant of the usual notion of ready simulation is employed to capture the refinement relation, which has been shown to be the largest precongruence satisfying some desired properties (Lüttgen and Vogler 2010).

Along the direction suggested by Lüttgen and Vogler in (Lüttgen and Vogler 2010), a process calculus called CLL is presented in (Zhang *et al.* 2011), which reconstructs their setting in pure process-algebraic style. Moreover, a sound and ground-complete proof system for CLL is provided. In effect, it gives an axiomatization of ready simulation in the presence of logic operators. However, due to lack of modal operators, CLL still does not afford describing abstract properties of systems. In (Zhu *et al.* 2013), CLL is enriched with temporal operators *always* and *unless* by two distinct approaches. Moreover, the connections between the resulting calculus and action-based computation tree logic are explored.

It is well known that recursive operations are fundamental mechanisms for representing objects with infinite behavior in terms of finite expressions (see, for instance (Bergstra *et al.* 2001)). Moreover, they are also powerful tools for integrating modal operators into process calculuses. For instance, the result obtained in (Zhu *et al.* 2013)

reveals that, under the mild assumption that the set of actions is finite, we can integrate temporal operators *always* and *unless* into CLL in a recursive manner. Such approach does not resort to any nonstandard operational operators, and hence is more succinct than one presented by Lüttgen and Vogler in (Lüttgen and Vogler 2011). However, to our knowledge, as yet no attempt has made to develop a general theory concerning recursive operations over LLTS and a few fundamental problems are still open. Since LLTS involves consideration of inconsistencies, it is far from straightforward to re-establish existent results concerning recursive operations in such framework. A solid effort is required, especially for handling inconsistencies. This paper intends to explore recursive operations over LLTS in pure process-algebraic style. We shall propose a process calculus named CLL_R, which is obtained by enriching CLL with recursive operations. The behavior theory of CLL_R and the uniqueness of solution for equations will be established.

The remainder of this paper is organized as follows. The next section recalls some related notions. Section 3 introduces SOS rules of CLL_R. In section 4, the existence and uniqueness of stable transition model for CLL_R is demonstrated, and a few of basic properties of the LTS associated with CLL_R are given. More further properties are considered in Section 5. In section 6, we shall show that the variant of ready simulation presented by Lüttgen and Vogler is precongruent in the presence of (nested) recursive operations. In section 7, a theorem on the uniqueness of solution for equations is obtained. Finally, a brief conclusion and discussion are given in Section 8.

2. Preliminaries

2.1. *Logic LTS and ready simulation*

This subsection will set up notations and briefly recall the notions of Logic LTS and ready simulation presented by Lüttgen and Vogler. For motivation behind these notions we refer the reader to (Lüttgen and Vogler 2007, 2010, 2011).

Let Act be the set of visible actions ranged over by letters a, b, etc., and let Act_τ denote $Act \cup \{\tau\}$ ranged over by α and β, where τ represents invisible actions. A labelled transition system (LTS) with a predicate is a quadruple $(P, Act_\tau, \longrightarrow, F)$, where P is a set of processes (states), $\longrightarrow \subseteq P \times Act_\tau \times P$ is the transition relation and $F \subseteq P$.

As usual, we write $p \xrightarrow{\alpha} q$ if $(p, \alpha, q) \in \longrightarrow$. q is said to be an α-derivative of p if $p \xrightarrow{\alpha} q$. We write $p \xrightarrow{\alpha}$ (or, $p \not\xrightarrow{\alpha}$) if $\exists q \in P.p \xrightarrow{\alpha} q$ ($\nexists q \in P.p \xrightarrow{\alpha} q$, respectively). Given a process p, the ready set $\{\alpha \in Act_\tau | p \xrightarrow{\alpha}\}$ of p is denoted by $\mathcal{I}(p)$. A state p is said to be stable if it cannot engage in any τ-transition, i.e., $p \not\xrightarrow{\tau}$. The list below contains some useful decorated transition relations:

$p \xrightarrow{\alpha}_F q$ iff $p \xrightarrow{\alpha} q$ and $p, q \notin F$.

$p \xRightarrow{\epsilon} q$ iff $p(\xrightarrow{\tau})^* q$, where $(\xrightarrow{\tau})^*$ is the transitive reflexive closure of $\xrightarrow{\tau}$.

$p \xRightarrow{\alpha} q$ iff $\exists r, s \in P.p \xRightarrow{\epsilon} r \xrightarrow{\alpha} s \xRightarrow{\epsilon} q$.

$p \xRightarrow{\gamma} |q$ iff $p \xRightarrow{\gamma} q \not\xrightarrow{\tau}$ with $\gamma \in Act_\tau \cup \{\epsilon\}$.

$p \xRightarrow{\epsilon}_F q$ iff there exists a sequence of $\tau-$labelled transitions from p to q such that all states along this sequence, including p and q, are not in F. The decorated transition $p \xRightarrow{\alpha}_F q$ may be defined similarly.

$p \stackrel{\epsilon}{\Longrightarrow}_F |q$ (or, $p \stackrel{\alpha}{\Longrightarrow}_F |q$) iff $p \stackrel{\epsilon}{\Longrightarrow}_F q$ ($p \stackrel{\alpha}{\Longrightarrow}_F q$, respectively) and q is stable.

Remark 2.1. Notice that some notations above are slightly different from ones adopted by Lüttgen and Vogler. In (Lüttgen and Vogler 2010, 2011) the notation $p \stackrel{\epsilon}{\Longrightarrow}|q$ (or, $p \stackrel{\alpha}{\Longrightarrow}|q$) has the same meaning as $p \stackrel{\epsilon}{\Longrightarrow}_F |q$ (respectively, $p \stackrel{\alpha}{\Longrightarrow}_F |q$) in this paper, while $p \stackrel{\epsilon}{\Longrightarrow} |q$ in this paper does not involve any requirement on consistency.

Definition 2.1 (Lüttgen and Vogler 2010). An LTS $(P, Act_\tau, \longrightarrow, F)$ is said to be an LLTS if, for each $p \in P$,
(LTS1) $p \in F$ if $\exists \alpha \in \mathcal{I}(p) \forall q \in P(p \stackrel{\alpha}{\longrightarrow} q$ implies $q \in F)$;
(LTS2) $p \in F$ if $\nexists q \in P.p \stackrel{\epsilon}{\Longrightarrow}_F |q$.

Here the predicate F is used to denote the set of all inconsistent states. Compared with usual LTSs, it is one distinguishing feature of LLTS that it involves consideration of inconsistencies. Roughly speaking, the motivation behind such consideration lies in dealing with inconsistencies caused by conjunctive composition. In the sequel, we shall use the phrase "*inconsistency predicate*" to refer to F. The condition (LTS1) formalizes the backward propagation of inconsistencies, and (LTS2) captures the intuition that divergence (i.e., infinite sequences of τ-transitions) should be viewed as catastrophic. For more intuitive ideas about inconsistency and motivation behind (LTS1) and (LTS2), the reader may refer to (Lüttgen and Vogler 2007, 2010).

Definition 2.2 (Lüttgen and Vogler 2010). An LTS $(P, Act_\tau, \longrightarrow, F)$ is said to be τ-pure if, for each $p \in P$, $p \stackrel{\tau}{\longrightarrow}$ implies $\nexists a \in Act.\ p \stackrel{a}{\longrightarrow}$.

Hence, for any state p in a τ-pure LTS, either $\mathcal{I}(p) = \{\tau\}$ or $\mathcal{I}(p) \subseteq Act$. Following (Lüttgen and Vogler 2010), this paper will focus on τ-pure LLTSs. In (Lüttgen and Vogler 2010, 2011), the notion of ready simulation below is adopted to capture the refinement relation, which is a variant of the usual notion of weak ready simulation.

Definition 2.3 (Ready simulation on LLTS). Let $(P, Act_\tau, \longrightarrow, F)$ be a LLTS. A relation $\mathcal{R} \subseteq P \times P$ is a stable ready simulation relation, if for any $(p, q) \in \mathcal{R}$ and $a \in Act$
(RS1) both p and q are stable;
(RS2) $p \notin F$ implies $q \notin F$;
(RS3) $p \stackrel{a}{\Longrightarrow}_F |p'$ implies $\exists q'.q \stackrel{a}{\Longrightarrow}_F |q'$ and $(p', q') \in \mathcal{R}$;
(RS4) $p \notin F$ implies $\mathcal{I}(p) = \mathcal{I}(q)$.

We say that p is stable ready simulated by q, in symbols $p \underset{\sim}{\sqsubset}_{RS} q$, if there exists a stable ready simulation relation $\mathcal{R}$ with $(p, q) \in \mathcal{R}$. Further, p is said to be ready simulated by q, written $p \sqsubseteq_{RS} q$, if

$$\forall p'(p \stackrel{\epsilon}{\Longrightarrow}_F |p' \text{ implies } \exists q'(q \stackrel{\epsilon}{\Longrightarrow}_F |q' \text{ and } p' \underset{\sim}{\sqsubset}_{RS} q')).$$

It is easy to see that $\underset{\sim}{\sqsubset}_{RS}$ is a stable ready simulation relation and both $\underset{\sim}{\sqsubset}_{RS}$ and $\sqsubseteq_{RS}$ are pre-order (i.e., reflexive and transitive). The equivalence relations induced by them are denoted by $\approx_{RS}$ and $=_{RS}$, respectively. That is,

$$\approx_{RS} \triangleq \underset{\sim}{\sqsubset}_{RS} \cap (\underset{\sim}{\sqsubset}_{RS})^{-1} \text{ and } =_{RS} \triangleq \sqsubseteq_{RS} \cap (\sqsubseteq_{RS})^{-1}.$$

2.2. *Transition system specification*

Structural Operational Semantics (SOS) is proposed by G. Plotkin in (Plotkin 1981), which adopts a syntax oriented view on operational semantics, and gives operational semantics in logical style. Transition System Specifications (TSSs), as presented by Groote and Vaandrager in (Groote 1992), are formalizations of SOS. This subsection recalls basic concepts related to TSS. For further information on this issue we refer the reader to (Aceto 2001; Bol and Groote 1996; Groote 1992).

Let V_{AR} be an infinite set of variables and Σ a signature. The set of Σ-terms over V_{AR}, denoted by $T(\Sigma, V_{AR})$, is the least set such that (I) $V_{AR} \subseteq T(\Sigma, V_{AR})$ and (II) if $f \in \Sigma$ and $t_1, \ldots, t_n \in T(\Sigma, V_{AR})$, then $f(t_1, \ldots, t_n) \in T(\Sigma, V_{AR})$, where n is the arity of f. $T(\Sigma, \emptyset)$ is abbreviated by $T(\Sigma)$, elements in $T(\Sigma)$ are called closed or ground terms.

A substitution σ is a mapping from V_{AR} to $T(\Sigma, V_{AR})$. As usual, a substitution σ may be lifted to a mapping $T(\Sigma, V_{AR}) \to T(\Sigma, V_{AR})$ by $\sigma(f(t_1, ..., t_n)) \triangleq f(\sigma(t_1), \ldots, \sigma(t_n))$ for any n-arity $f \in \Sigma$ and $t_1, \ldots, t_n \in T(\Sigma, V_{AR})$. A substitution is said to be closed if it maps all variables to ground terms.

A TSS is a quadruple $\mathcal{P} = (\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$, where Σ is a signature, $\mathbb{A}$ is a set of labels, $\mathbb{P}$ is a set of predicate symbols and $\mathbb{R}$ is a set of rules. Positive literals are all expressions of the form $t \xrightarrow{a} t'$ or tP, while negative literals are all expressions of the form $t \not\xrightarrow{a}$ or $t\neg P$, where $t, t' \in T(\Sigma, V_{AR})$, $a \in \mathbb{A}$ and $P \in \mathbb{P}$. A literal is a positive or negative literal, and φ, ψ, χ are used to range over literals. A literal is said to be ground or closed if all terms occurring in it are ground. A rule $r \in \mathbb{R}$ has the form like $\frac{H}{C}$, where H, the premises of the rule r, denoted as $prem(r)$, is a set of literals, and C, the conclusion of the rule r, denoted as $conc(r)$, is a positive literal. Furthermore, we write $pprem(r)$ for the set of positive premises of r and $nprem(r)$ for the set of negative premises of r. A rule r is said to be positive if $nprem(r) = \emptyset$. A TSS is said to be positive if it has only positive rules. Given a substitution σ and a rule $r \in \mathbb{R}$, $\sigma(r)$ is the rule obtained from r by replacing each variable in r by its σ-image, that is, $\sigma(r) \triangleq \frac{\{\sigma(\varphi) | \varphi \in prem(r)\}}{\sigma(conc(r))}$. Moreover, if σ is closed then $\sigma(r)$ is said to be a ground instance of r.

Definition 2.4 (Proof in positive TSS). Let $\mathcal{P} = (\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$ be a positive TSS. A proof of a closed positive literal ψ from $\mathcal{P}$ is a well-founded, upwardly branching tree, whose nodes are labelled with closed positive literals, such that
— the root is labelled with ψ,
— if χ is the label of a node q and $\{\chi_i : i \in I\}$ is the set of labels of the nodes directly above q, then there is a rule $\{\varphi_i : i \in I\}/\varphi$ in $\mathbb{R}$ and a closed substitution σ such that $\chi = \sigma(\varphi)$ and $\chi_i = \sigma(\varphi_i)$ for each $i \in I$.

If a proof of ψ from $\mathcal{P}$ exists, then ψ is said to be provable from $\mathcal{P}$, in symbols $\mathcal{P} \vdash \psi$.

A natural and simple method of describing the operational nature of closed terms is in terms of LTSs. Given a TSS, an important problem is how to associate LTS with any given closed terms. For positive TSS, the answer is straightforward. However, this problem is far from trivial for TSS containing negative premises. The notions of stable model and stratification of TSS play an important role in dealing with this issue. The remainder of this subsection intends to recall these notions briefly.

Given a TSS $\mathcal{P} = (\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$, a transition model M is a subset of $Tr(\Sigma, \mathbb{A}) \cup Pred(\Sigma, \mathbb{P})$, where $Tr(\Sigma, \mathbb{A}) = T(\Sigma) \times \mathbb{A} \times T(\Sigma)$ and $Pred(\Sigma, \mathbb{P}) = T(\Sigma) \times \mathbb{P}$, elements (t, a, t') and (t, P) in M are written as $t \xrightarrow{a} t'$ and tP respectively. A positive closed literal ψ holds in M or ψ is valid in M, in symbols $M \models \psi$, if $\psi \in M$. A negative closed literal $t \stackrel{a}{\not\longrightarrow}$ (or, $t\neg P$) holds in M, in symbols $M \models t \stackrel{a}{\not\longrightarrow}$ ($M \models t\neg P$, respectively), if there is no t' such that $t \xrightarrow{a} t' \in M$($tP \notin M$, respectively). For a set of closed literals Ψ, we write $M \models \Psi$ iff $M \models \psi$ for each $\psi \in \Psi$. M is said to be a model of $\mathcal{P}$ if, for each $r \in \mathbb{R}$ and $\sigma : V_{AR} \longrightarrow T(\Sigma)$, we have $M \models conc(\sigma(r))$ whenever $M \models prem(\sigma(r))$. M is said to be supported by $\mathcal{P}$ if, for each $\psi \in M$, there exists $r \in \mathbb{R}$ and $\sigma : V_{AR} \longrightarrow T(\Sigma)$ such that $M \models prem(\sigma(r))$ and $conc(\sigma(r)) = \psi$. M is said to be a supported model of $\mathcal{P}$ if M is supported by $\mathcal{P}$ and M is a model of $\mathcal{P}$.

Definition 2.5 (Aceto 2001; Bol and Groote 1996). Let $\mathcal{P} = (\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$ be a TSS and α an ordinal number. A function $S : Tr(\Sigma, \mathbb{A}) \cup Pred(\Sigma, \mathbb{P}) \longrightarrow \alpha$ is said to be a stratification of $\mathcal{P}$ if, for every rule $r \in \mathbb{R}$ and every substitution $\sigma : V_{AR} \longrightarrow T(\Sigma)$, the following conditions hold.
(1) $S(\psi) \leq S(conc(\sigma(r)))$ for each $\psi \in pprem(\sigma(r))$,
(2) $S(tP) < S(conc(\sigma(r)))$ for each $t\neg P \in nprem(\sigma(r))$, and
(3) $S(t \xrightarrow{a} t') < S(conc(\sigma(r)))$ for each $t' \in T(\Sigma)$ and $t \stackrel{a}{\not\longrightarrow} \in nprem(\sigma(r))$.

A TSS is said to be stratified iff there exists a stratification function for it.

Definition 2.6 (Bol and Groote 1996; Gelfond and Lifchitz 1988). Let $\mathcal{P} = (\Sigma, \mathbb{A}, \mathbb{P}, \mathbb{R})$ be a TSS and M a transition model. M is said to be a stable transition model for $\mathcal{P}$ if

$$M = M_{Strip(\mathcal{P},M)},$$

where $Strip(\mathcal{P}, M)$ is the TSS $(\Sigma, \mathbb{A}, \mathbb{P}, Strip(\mathbb{R}, M))$ with

$$Strip(\mathbb{R}, M) \triangleq \left\{ \frac{pprem(r)}{conc(r)} | \quad r \in \mathbb{R}_{ground} \text{ and } M \models nprem(r) \right\},$$

where $\mathbb{R}_{ground}$ denotes the set of all ground instances of rules in $\mathbb{R}$, and $M_{Strip(\mathcal{P},M)}$ is the least transition model of the positive TSS $Strip(\mathcal{P}, M)$.

As is well known, stable models are supported models and each stratified TSS $\mathcal{P}$ has a unique stable model (Bol and Groote 1996), moreover, such stable model does not depend on particular stratification function (Groote 1993).

3. Syntax and SOS rules of CLL$_R$

The calculus CLL$_R$ is obtained from CLL by enriching it with recursive operations. Following (Baeten and Bravetti 2008), this paper adopts the notation $\langle X|E(V)\rangle$ to denote recursive operations, which encompasses both the CCS operator $recX.t$ and standard way of expressing recursion in ACP. Formally, the terms in CLL$_R$ are defined by BNF below:

$$t ::= 0 \mid \bot \mid (\alpha.t) \mid (t\Box t) \mid (t \wedge t) \mid (t \vee t) \mid (t \parallel_A t) \mid X \mid \langle X|E(V)\rangle$$

where $X \in V_{AR}$, $\alpha \in Act_\tau$, $A \subseteq Act$, $V \subseteq V_{AR}$ and $\langle X|E(V)\rangle$ is a recursive operation with recursive specification $E(V)$ and initial variable $X \in V$. A recursive specification $E(V)$ is a nonempty finite set of equations that contains precisely one equation with the form like $X = t_X$ for each $X \in V$, where t_X itself is a term in CLL_R. In the sequel, we often abbreviate $\langle X|E(V)\rangle$ to $\langle X|E\rangle$ and denote $\langle X|\{X = t_X\}\rangle$ briefly by $\langle X|X = t_X\rangle$.

In the following, given a term $\langle X|E\rangle$ and variable Y, the phrase "Y occurs in $\langle X|E\rangle$" means that Y occurs in t_Z for some $Z = t_Z \in E$. Moreover, the scope of a recursive operation $\langle X|E\rangle$ exactly consists of all t_Z with $Z = t_Z \in E$. An occurrence of a variable X in a given t is said to be free if it does not occur in the scope of any recursive operation $\langle Y|E(V)\rangle$ with $X \in V$. A variable X in term t is said to be a free variable if all occurrences of X in t are free, otherwise X is said to be a recursive variable in t.

Convention 3.1. Throughout this paper, as usual, we make the assumption that recursive variables are distinct from each other. That is, for any two recursive specifications $E(V_1)$ and $E'(V_2)$ we have $V_1 \cap V_2 = \emptyset$. Moreover, we will tacitly restrict our attention to terms where no recursive variable has free occurrences. For example we will not consider terms such as $X\Box\langle X|X = a.X\rangle$ because this term could be replaced by the clear term $X\Box\langle Y|Y = a.Y\rangle$ with the same meaning.

On account of the above convention, given a term t, the set $FV(t)$ of all free variables of t may be defined recursively as:

— $FV(X) = \{X\}$; $FV(0) = FV(\bot) = \emptyset$; $FV(\alpha.t) = FV(t)$;
— $FV(t_1 \odot t_2) = FV(t_1) \cup FV(t_2)$ with $\odot \in \{\vee, \wedge, \Box, \|_A\}$;
— $FV(\langle Y|E(V)\rangle) = \bigcup_{Z=t_Z\in E} FV(t_Z) - V$.

As usual, a term t is said to be closed if $FV(t) = \emptyset$. The set of all closed terms of CLL_R is denoted as $T(\Sigma_{\mathrm{CLL}_R})$. In the following, a term is said to be a *process* iff it is closed. Unless noted, we use p, q, r to represent processes. We shall always use $t_1 \equiv t_2$ to mean that expressions t_1 and t_2 are syntactically identical. In particular, $\langle Y|E\rangle \equiv \langle Y'|E'\rangle$ means that $Y \equiv Y'$ and for any Z and t_Z, $Z = t_Z \in E$ iff $Z = t_Z \in E'$.

Definition 3.1. For any recursive specification $E(V)$ and term t, we define $\langle t|E\rangle$ to be $t\{\langle X|E\rangle/X : X \in V\}$, that is, $\langle t|E\rangle$ is obtained from t by simultaneously replacing all free occurrences of each $X(\in V)$ by $\langle X|E\rangle$.

For example, consider $t \equiv X\Box a.\langle Y|Y = X\ \Box Y\rangle$ and $E(\{X\}) = \{X = t_X\}$ then $\langle t|E\rangle \equiv \langle X|X = t_X\rangle\Box a.\langle Y|Y = \langle X|X = t_X\rangle\Box Y\rangle$. In particular, for any recursive specification $E(V)$ and $t \equiv X$, $\langle t|E\rangle \equiv \langle X|E\rangle$ whenever $X \in V$ and $\langle t|E\rangle \equiv X$ if $X \notin V$.

Given a term t, the set $SubT(t)$ of all subterms of t may be defined recursively as:

— $SubT(0) = \{0\}$; $SubT(\bot) = \{\bot\}$; $SubT(X) = \{X\}$; $SubT(\alpha.t) = SubT(t) \cup \{\alpha.t\}$;
— $SubT(t_1 \odot t_2) = SubT(t_1) \cup SubT(t_2) \cup \{t_1 \odot t_2\}$ with $\odot \in \{\vee, \wedge, \Box, \|_A\}$;
— $SubT(\langle Y|E\rangle) = \bigcup_{Z=t_Z\in E} SubT(t_Z) \cup \{\langle Y|E\rangle\}$.

As usual, an occurrence of X in t is said to be strongly (or, weakly) guarded if such occurrence is within some subexpression $a.t_1$ with $a \in Act$ ($\tau.t_1$ or $t_1 \vee t_2$, respectively). A variable X is strongly (or, weakly) guarded in t if each occurrence of X is strongly (weakly, respectively) guarded. Notice that, since the first move of $r \vee s$ is a τ-labelled

$$(Ra_1)\,\frac{-}{\alpha.x_1 \xrightarrow{\alpha} x_1} \qquad (Ra_2)\,\frac{x_1 \xrightarrow{a} y_1,\ x_2 \overset{\tau}{\not\longrightarrow}}{x_1 \Box x_2 \xrightarrow{a} y_1}$$

$$(Ra_3)\,\frac{x_1 \overset{\tau}{\not\longrightarrow},\ x_2 \xrightarrow{a} y_2}{x_1 \Box x_2 \xrightarrow{a} y_2} \qquad (Ra_4)\,\frac{x_1 \xrightarrow{\tau} y_1}{x_1 \Box x_2 \xrightarrow{\tau} y_1 \Box x_2}$$

$$(Ra_5)\,\frac{x_2 \xrightarrow{\tau} y_2}{x_1 \Box x_2 \xrightarrow{\tau} x_1 \Box y_2} \qquad (Ra_6)\,\frac{x_1 \xrightarrow{a} y_1,\ x_2 \xrightarrow{a} y_2}{x_1 \wedge x_2 \xrightarrow{a} y_1 \wedge y_2}$$

$$(Ra_7)\,\frac{x_1 \xrightarrow{\tau} y_1}{x_1 \wedge x_2 \xrightarrow{\tau} y_1 \wedge x_2} \qquad (Ra_8)\,\frac{x_2 \xrightarrow{\tau} y_2}{x_1 \wedge x_2 \xrightarrow{\tau} x_1 \wedge y_2}$$

$$(Ra_9)\,\frac{-}{x_1 \vee x_2 \xrightarrow{\tau} x_1} \qquad (Ra_{10})\,\frac{-}{x_1 \vee x_2 \xrightarrow{\tau} x_2}$$

$$(Ra_{11})\,\frac{x_1 \xrightarrow{\tau} y_1}{x_1 \parallel_A x_2 \xrightarrow{\tau} y_1 \parallel_A x_2} \qquad (Ra_{12})\,\frac{x_2 \xrightarrow{\tau} y_2}{x_1 \parallel_A x_2 \xrightarrow{\tau} x_1 \parallel_A y_2}$$

$$(Ra_{13})\,\frac{x_1 \xrightarrow{a} y_1,\ x_2 \overset{\tau}{\not\longrightarrow}}{x_1 \parallel_A x_2 \xrightarrow{a} y_1 \parallel_A x_2}(a \notin A) \qquad (Ra_{14})\,\frac{x_1 \overset{\tau}{\not\longrightarrow},\ x_2 \xrightarrow{a} y_2}{x_1 \parallel_A x_2 \xrightarrow{a} x_1 \parallel_A y_2}(a \notin A)$$

$$(Ra_{15})\,\frac{x_1 \xrightarrow{a} y_1,\ x_2 \xrightarrow{a} y_2}{x_1 \parallel_A x_2 \xrightarrow{a} y_1 \parallel_A y_2}(a \in A) \qquad (Ra_{16})\,\frac{\langle t_X|E\rangle \xrightarrow{\alpha} y}{\langle X|E\rangle \xrightarrow{\alpha} y}(X = t_X \in E)$$

Table 1. Operational rules

transition (see Table 1), which is independent of r and s, any occurrence of X in $r \vee s$ is treated as being weakly guarded. A recursive specification $E(V)$ is said to be guarded if for each $X \in V$ and $Z = t_Z \in E$, each occurrence of X in t_Z is (weakly or strongly) guarded.

Convention 3.2. It is well known that unguarded processes cause many problems in many aspects of the theory (Milner 1983) and unguarded recursion is incompatible with negative rules (Bloom 1994). As usual, this paper will focus on guarded recursive specifications. That is, we assume that all recursive specifications considered in the remainder of this paper are guarded.

We now provide SOS rules to specify the behavior of processes (i.e., closed terms) formally. All SOS rules are divided into two parts: operational and predicate rules.

Operational rules $Ra_i (1 \leq i \leq 16)$ are listed in Table 1, where $a \in Act$, $\alpha \in Act_\tau$ and $A \subseteq Act$. Negative premises in rules Ra_2, Ra_3, Ra_{13} and Ra_{14} give τ-transition precedence over transitions labelled with visible actions, which guarantees that the transition model of CLL_R is τ-pure. Rules Ra_9 and Ra_{10} illustrate that the operational aspect of $t_1 \vee t_2$ is same as internal choice in usual process calculus. The rule Ra_6 reflects that conjunction operator is a synchronous product for visible transitions. The operational rules about other operators are usual.

Predicate rules in Table 2 specify the inconsistency predicate F. Although both 0 and $\perp$ have empty behavior, they represent different processes. The rule Rp_1 says that $\perp$ is

$$(Rp_1)\frac{-}{\bot F} \qquad (Rp_2)\frac{x_1F}{\alpha.x_1F}$$

$$(Rp_3)\frac{x_1F, x_2F}{x_1 \vee x_2F} \qquad (Rp_4)\frac{x_1F}{x_1\Box x_2F}$$

$$(Rp_5)\frac{x_2F}{x_1\Box x_2F} \qquad (Rp_6)\frac{x_1F}{x_1 \parallel_A x_2F}$$

$$(Rp_7)\frac{x_2F}{x_1 \parallel_A x_2F} \qquad (Rp_8)\frac{x_1F}{x_1 \wedge x_2F}$$

$$(Rp_9)\frac{x_2F}{x_1 \wedge x_2F} \qquad (Rp_{10})\frac{x_1 \xrightarrow{a} y_1, x_2 \not\xrightarrow{a}, x_1 \wedge x_2 \not\xrightarrow{\tau}}{x_1 \wedge x_2F}$$

$$(Rp_{11})\frac{x_1 \not\xrightarrow{a}, x_2 \xrightarrow{a} y_2, x_1 \wedge x_2 \not\xrightarrow{\tau}}{x_1 \wedge x_2F} \qquad (Rp_{12})\frac{x_1 \wedge x_2 \xrightarrow{\alpha} z, \{yF : x_1 \wedge x_2 \xrightarrow{\alpha} y\}}{x_1 \wedge x_2F}$$

$$(Rp_{13})\frac{\{yF : x_1 \wedge x_2 \overset{\epsilon}{\Longrightarrow} |y\}}{x_1 \wedge x_2F} \qquad (Rp_{14})\frac{\langle t_X|E\rangle F}{\langle X|E\rangle F}(X = t_X \in E)$$

$$(Rp_{15})\frac{\{yF : \langle X|E\rangle \overset{\epsilon}{\Longrightarrow} |y\}}{\langle X|E\rangle F}$$

Table 2. Predicate rules

inconsistent. Thus $\bot$ cannot be implemented. While 0 is consistent, which is an implementable process. The rule Rp_3 reflects that if both two disjunctive parts are inconsistent then so is the disjunction. Rules $Rp_4 - Rp_9$ describe the system design strategy that if one part is inconsistent, then so is the whole composition. The rules Rp_{10} and Rp_{11} reveal that a stable conjunction is inconsistent if its conjuncts have distinct ready sets.

The rule Rp_{12} formalizes (LTS1) in Def. 2.1 for processes of the form $x_1 \wedge x_2$. Although the universal quantifier symbol does not occur in Rp_{12} explicitly, it is not difficult to see that the premise of Rp_{12} involves universal quantifier in spirit. Analogous to CLL (Zhang *et al.* 2011), we may adopt the method presented in (Verhoef 95) to avoid this. In detail, for each $\alpha \in Act_\tau$, the auxiliary predicate F_α is added to CLL_R and the rule Rp_{12} is replaced by two rules below

$$(Rp_{12-1})\frac{x_1 \wedge x_2 \xrightarrow{\alpha} y, y\neg F}{x_1 \wedge x_2F_\alpha} \qquad (Rp_{12-2})\frac{x_1 \wedge x_2 \xrightarrow{\alpha} y, x_1 \wedge x_2\neg F_\alpha}{x_1 \wedge x_2F}.$$

Intuitively, pF_α says that p has a consistent α-derivative. Similar to Rp_{12}, these two rules also capture (LTS1). However, it is easy to see that, due to Rp_{12-1} and Rp_{12-2}, the stratifying function does not exist for the resulting calculus. By means of technique so-called *positive after reduction* (Bol and Groote 1996), we can also get its stable transition model as done in (Zhang *et al.* 2011). Moreover, such stable transition model coincides with M_{CLL_R} obtained in the next section. To avoid cumbersome reduction procedure, our current system employs Rp_{12} instead of Rp_{12-1} and Rp_{12-2}.

Rules Rp_{13} and Rp_{15} are used to capture (LTS2) in Def. 2.1, which are the abbreviation

of the rules with the format below

$$\frac{\{yF : \exists y_0, y_1, \ldots, y_n(z \equiv y_0 \xrightarrow{\tau} y_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} y_n \equiv y \text{ and } y \not\xrightarrow{\tau})\}}{zF}$$

with $z \equiv x_1 \wedge x_2$ or $\langle X|E\rangle$. Intuitively, these two rules say that if all stable τ-descendants of z are inconsistent, then z itself is inconsistent. Notice that, especially for readers who are familiar with notations used in (Lüttgen and Vogler 2010), the transition relation $\overset{\epsilon}{\Longrightarrow}|$ occurring in these two rules does not involve any requirement on consistency (see Remark 2.1 and notations above it).

Since the behavior of any process in CLL is finite, each process can reach a stable state, and rules $Rp_1 - Rp_{12}$ suffice to capture the inconsistency predicate F. In particular, these rules guarantee that the LTS associated with CLL satisfies (LTS1) and (LTS2) in Def. 2.1 (Zhang *et al.* 2011). However, for CLL_R, $Rp_1 - Rp_{12}$ are insufficient even if the usual rule for recursive operations (i.e. Rp_{14}) is added. For instance, consider processes $q \equiv \langle X|X = \tau.X\rangle$ and $p \equiv \langle X|X = X \vee 0\rangle \wedge a.0$, it is not difficult to see that neither qF nor pF can be inferred by using only rules $Rp_1 - Rp_{12}$ and Rp_{14}, however, both p and q should be inconsistent due to (LTS2). Fortunately, an inference of pF (or, qF) is at hand by admitting the rule Rp_{13} (Rp_{15}, respectively).

Summarizing, the TSS for CLL_R is $\mathcal{P}_{\text{CLL}_R} = (\Sigma_{\text{CLL}_R}, Act_\tau, \mathbb{P}_{\text{CLL}_R}, \mathbb{R}_{\text{CLL}_R})$, where

— $\Sigma_{\text{CLL}_R} = \{\Box, \wedge, \vee, 0, \bot\} \cup \{\alpha.() : \alpha \in Act_\tau\} \cup \{\|_A\colon A \subseteq Act\} \cup \{\langle X|E(V)\rangle :$ $E(V)$ is a guarded recursive specification with $X \in V\}$,
— $\mathbb{P}_{\text{CLL}_R} = \{F\}$, and
— $\mathbb{R}_{\text{CLL}_R} = \{Ra_1, \ldots, Ra_{16}\} \cup \{Rp_1, \ldots, Rp_{15}\}$.

4. Stable transition model of $\mathcal{P}_{\text{CLL}_R}$

This section will consider the well-definedness of the TSS $\mathcal{P}_{\text{CLL}_R}$ (i.e., the existence and uniqueness of the stable model of $\mathcal{P}_{\text{CLL}_R}$) and provide a few basic properties of the LTS associated with $\mathcal{P}_{\text{CLL}_R}$

In order to demonstrate that $\mathcal{P}_{\text{CLL}_R}$ has a unique stable model, it is enough to give a stratification function of $\mathcal{P}_{\text{CLL}_R}$. To this end, a few preliminary notations are introduced. Given a term t, the degree of t, denoted by $|t|$, is inductively defined as

— $|0| = |\bot| = |\langle X|E\rangle| \triangleq 1$;
— $|t_1 \odot t_2| \triangleq |t_1| + |t_2| + 1$ for each $\odot \in \{\wedge, \Box, \vee, \|_A\}$;
— $|\alpha.t| \triangleq |t| + 1$ with $\alpha \in Act_\tau$.

The function $G : T(\Sigma_{\text{CLL}_R}) \longrightarrow \mathbb{N}$ is defined by

— $G(\langle X|E\rangle) \triangleq 1$;
— $G(0) = G(\bot) = G(\alpha.t) = G(t_1 \vee t_2) \triangleq 0$ with $\alpha \in Act_\tau$;
— $G(t_1 \odot t_2) \triangleq G(t_1) + G(t_2)$ for each $\odot \in \{\wedge, \Box, \|_A\}$.

Further, the function $S_{\mathcal{P}_{\text{CLL}_R}}$ from $Tr(\Sigma_{\text{CLL}_R}, Act_\tau) \cup Pred(\Sigma_{\text{CLL}_R}, \mathbb{P}_{\text{CLL}_R})$ to $\omega \times 2+1$ is given below, where ω is the initial limit ordinal,

— $S_{\mathcal{P}_{\text{CLL}_R}}(t \xrightarrow{\alpha} t') \triangleq G(t) \times \omega + |t|$;
— $S_{\mathcal{P}_{\text{CLL}_R}}(tF) \triangleq \omega \times 2$.

Since each recursive specification is assumed to be guarded (see, Convention 3.2), it is not difficult to check that this function $S_{\mathcal{P}_{\mathrm{CLL}_R}}$ is a stratification of $\mathcal{P}_{\mathrm{CLL}_R}$. Moreover, since each stratified TSS has a unique stable model (Bol and Groote 1996), $\mathcal{P}_{\mathrm{CLL}_R}$ has a unique stable transition model. From now on, we use M_{CLL_R} to denote such stable model.

Definition 4.1. The LTS associated with CLL_R, in symbols $LTS(\mathrm{CLL}_R)$, is the quadruple $(T(\Sigma_{\mathrm{CLL}_R}), Act_\tau, \longrightarrow_{\mathrm{CLL}_R}, F_{\mathrm{CLL}_R})$, where
— $p \xrightarrow{\alpha}_{\mathrm{CLL}_R} p'$ iff $p \xrightarrow{\alpha} p' \in M_{\mathrm{CLL}_R}$;
— $p \in F_{\mathrm{CLL}_R}$ iff $pF \in M_{\mathrm{CLL}_R}$.

Therefore, $p \xrightarrow{\alpha}_{\mathrm{CLL}_R} p'$ (or, $p \in F_{\mathrm{CLL}_R}$) if and only if $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \xrightarrow{\alpha} p'$ (pF, respectively) for any processes p, p' and $\alpha \in Act_\tau$. This allows us to proceed by induction on the depth of inferences when demonstrating propositions concerning $\longrightarrow_{\mathrm{CLL}_R}$ and F_{CLL_R}.

Convention 4.1. For the sake of convenience, in the remainder of this paper, we shall omit the subscript in labelled transition relations $\xrightarrow{\alpha}_{\mathrm{CLL}_R}$, that is, we shall use $\xrightarrow{\alpha}$ to denote transition relation in $LTS(\mathrm{CLL}_R)$. Thus, the notation $\xrightarrow{\alpha}$ has double utility: predicate symbols in the TSS $\mathcal{P}_{\mathrm{CLL}_R}$ and labelled transition relations on processes in $LTS(\mathrm{CLL}_R)$. However, it usually does not lead to confusion in a given context. Similarly, the notation F_{CLL_R} will be abbreviated to F. Hence the symbol F is overloaded, predicate symbol in the TSS $\mathcal{P}_{\mathrm{CLL}_R}$ and the set of all inconsistent processes within $LTS(\mathrm{CLL}_R)$, in each case the context of use will allow us to make the distinction.

In the following, we intend to provide a number of simple properties of $LTS(\mathrm{CLL}_R)$. In particular, we will show that $LTS(\mathrm{CLL}_R)$ is a τ-pure LLTS.

Lemma 4.1. Let p and q be any two processes.
(1) $p \vee q \in F$ iff $p, q \in F$.
(2) $\alpha.p \in F$ iff $p \in F$ for each $\alpha \in Act_\tau$.
(3) $p \odot q \in F$ iff either $p \in F$ or $q \in F$ with $\odot \in \{\Box, \|_A\}$.
(4) Either $p \in F$ or $q \in F$ implies $p \wedge q \in F$.
(5) $0 \notin F$ and $\bot \in F$.
(6) $\langle X | X = \tau.X \rangle \in F$.
(7) If $\forall q(p \overset{\epsilon}{\Longrightarrow} |q$ implies $q \in F)$ then $p \in F$.
(8) $\langle X|E \rangle \in F$ iff $\langle t_X|E \rangle \in F$ for each X with $X = t_X \in E$.

Proof. Items (1) - (6) are straightforward. For item (7), it proceeds by induction on p, in particular, for the case where p is of the format $p_1 \wedge p_2$ (or $\langle X|E \rangle$), the conclusion immediately follows due to Rp_{13} (respectively, Rp_{15}).

For item (8), the implication from right to left is straightforward. The argument of the converse implication splits into two cases based on the last rule applied in the proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash \langle X|E \rangle F$. If Rp_{14} is the last rule then the proof is trivial. For another case where Rp_{15} is used, it is also straightforward by applying item (7) in this lemma and the fact that $\langle X|E \rangle \xrightarrow{\tau} r$ iff $\langle t_X|E \rangle \xrightarrow{\tau} r$ for any r. □

The notion of τ-pure is a technique constraint for LLTSs (Lüttgen and Vogler 2007, 2010). The result below shows that $LTS(\mathrm{CLL}_R)$ is indeed τ-pure.

Theorem 4.1. $LTS(\mathrm{CLL}_R)$ is τ-pure.

Proof. Suppose $p \xrightarrow{\tau}$. Hence $p \xrightarrow{\tau} q$ for some q. Then the lemma would be established by proving that $p \not\xrightarrow{a}$ for any $a \in Act$. It is straightforward by induction on the depth of the proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \xrightarrow{\tau} q$. □

In order to prove that $LTS(\mathrm{CLL}_R)$ is a LLTS, the result below is needed. Its converse is an instance of (LTS1) with $\alpha = \tau$, and hence also holds by Theorem 4.2.

Lemma 4.2. For any process p with $\tau \in \mathcal{I}(p)$, if $p \in F$ then $\forall q(p \xrightarrow{\tau} q$ implies $q \in F)$.

Proof. Suppose $p \xrightarrow{\tau} q$. We may prove $q \in F$ by induction on the depth of the proof tree $\mathcal{T}$ of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \xrightarrow{\tau} q$. It proceeds by distinguishing different cases based on the form of p. Here we handle only three cases as samples.

Case 1 $p \equiv p_1 \Box p_2$.

W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{p_1 \xrightarrow{\tau} p_1'}{p_1 \Box p_2 \xrightarrow{\tau} p_1' \Box p_2}$. Hence $q \equiv p_1' \Box p_2$. Since $p \in F$, by Lemma 4.1(3), either $p_1 \in F$ or $p_2 \in F$. If $p_2 \in F$ then it immediately follows from Lemma 4.1(3) that $q \equiv p_1' \Box p_2 \in F$. If $p_1 \in F$ then $p_1' \in F$ by induction hypothesis (IH, for short). Hence $p_1' \Box p_2 \in F$, as desired.

Case 2 $p \equiv \langle X|E\rangle$.

The last rule applied in $\mathcal{T}$ is $\frac{\langle t_X|E\rangle \xrightarrow{\tau} q}{\langle X|E\rangle \xrightarrow{\tau} q}$ with $X = t_X \in E$. Since $p \in F$, by Lemma 4.1(8), we have $\langle t_X|E\rangle \in F$. Then $q \in F$ by applying IH.

Case 3 $p \equiv p_1 \wedge p_2$.

W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{p_1 \xrightarrow{\tau} p_1'}{p_1 \wedge p_2 \xrightarrow{\tau} p_1' \wedge p_2}$. Hence $q \equiv p_1' \wedge p_2$. In the following, we intend to show $q \in F$ by distinguishing four cases based on the last rule applied in the inference of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_1 \wedge p_2 F$.

Case 3.1 $\frac{p_1 F}{p_1 \wedge p_2 F}$ or $\frac{p_2 F}{p_1 \wedge_2 F}$.
Similar to Case 1, omitted.

Case 3.2 $\frac{p_1 \xrightarrow{a} r,\, p_2 \not\xrightarrow{a},\, p_1 \wedge p_2 \not\xrightarrow{\tau}}{p_1 \wedge p_2 F}$ or $\frac{p_1 \not\xrightarrow{a},\, p_2 \xrightarrow{a} r,\, p_1 \wedge p_2 \not\xrightarrow{\tau}}{p_1 \wedge p_2 F}$.
This case is impossible because of $\tau \in \mathcal{I}(p_1 \wedge p_2)$.

Case 3.3 $\frac{p_1 \wedge p_2 \xrightarrow{\alpha} r,\, \{r' F : p_1 \wedge p_2 \xrightarrow{\alpha} r'\}}{p_1 \wedge p_2 F}$.
Since $LTS(\mathrm{CLL}_R)$ is τ-pure and $p_1 \wedge p_2 \xrightarrow{\tau}$, we have $\alpha = \tau$. Hence $q \in F$ immediately.

Case 3.4 $\frac{\{rF : p_1 \wedge p_2 \overset{\epsilon}{\Longrightarrow} |r\}}{p_1 \wedge p_2 F}$.

Assume $q \equiv p_1' \wedge p_2 \stackrel{\epsilon}{\Longrightarrow} |r'$. Thus $r' \in F$ due to $p \stackrel{\tau}{\longrightarrow} p_1' \wedge p_2 \stackrel{\epsilon}{\Longrightarrow} |r'$. Hence $p_1' \wedge p_2 \in F$ by applying the rule Rp_{13}. □

Now we are ready to show that $LTS(\mathrm{CLL}_R)$ is a LLTS.

Theorem 4.2. $LTS(\mathrm{CLL}_R)$ is a LLTS.

Proof. **(LTS1)** Suppose $\alpha \in \mathcal{I}(p)$ and $\forall r(p \stackrel{\alpha}{\longrightarrow} r$ implies $r \in F)$. Then $p \stackrel{\alpha}{\longrightarrow} q$ for some q. To complete the proof, we intend to show $p \in F$. It proceeds by induction on the depth of the proof tree $\mathcal{T}$ of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \stackrel{\alpha}{\longrightarrow} q$. We distinguish different cases based on the form of p. In particular, the proof for the case $p \equiv p_1 \wedge p_2$ is immediate by the rule Rp_{12}. In the following, we give the proof for the case $p \equiv p_1 \parallel_A p_2$, the other cases are left to the reader. The argument splits into two cases depending on α.

Case 1 $\alpha = \tau$.

W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{p_1 \stackrel{\tau}{\longrightarrow} p_1'}{p_1 \parallel_A p_2 \stackrel{\tau}{\longrightarrow} p_1' \parallel_A p_2}$. Thus $q \equiv p_1' \parallel_A p_2$. If $p_2 \in F$ then $p_1 \parallel_A p_2 \in F$ comes from Lemma 4.1(3) at once. For another case $p_2 \notin F$, it is not difficult to see that each τ-derivative of p_1 is inconsistent, that is $\forall p_1''(p_1 \stackrel{\tau}{\longrightarrow} p_1''$ implies $p_1'' \in F)$. Hence $p_1 \in F$ by IH. Therefore it follows from Lemma 4.1(3) that $p_1 \parallel_A p_2 \in F$, as desired.

Case 2 $\alpha \in Act$.

In such situation, the last rule applied in $\mathcal{T}$ has one of the following three formats: (1) $\frac{p_1 \stackrel{\alpha}{\longrightarrow} p_1', p_2 \stackrel{\tau}{\not\longrightarrow}}{p_1 \parallel_A p_2 \stackrel{\alpha}{\longrightarrow} p_1' \parallel_A p_2}(\alpha \notin A)$; (2) $\frac{p_2 \stackrel{\alpha}{\longrightarrow} p_2', p_1 \stackrel{\tau}{\not\longrightarrow}}{p_1 \parallel_A p_2 \stackrel{\alpha}{\longrightarrow} p_1 \parallel_A p_2'}(\alpha \notin A)$; (3) $\frac{p_1 \stackrel{\alpha}{\longrightarrow} p_1', p_2 \stackrel{\alpha}{\longrightarrow} p_2'}{p_1 \parallel_A p_2 \stackrel{\alpha}{\longrightarrow} p_1' \parallel_A p_2'}(\alpha \in A)$.

We consider only (3), the other two may be handled in a similar manner as the case $\alpha = \tau$. Since $\forall r(p_1 \parallel_A p_2 \stackrel{\alpha}{\longrightarrow} r$ implies $r \in F)$, by Lemma 4.1(3), it is easy to see that either $\forall r(p_1 \stackrel{\alpha}{\longrightarrow} r$ implies $r \in F)$ or $\forall r(p_2 \stackrel{\alpha}{\longrightarrow} r$ implies $r \in F)$. On the other hand, due to $\alpha \in \mathcal{I}(p_1)$ and $\alpha \in \mathcal{I}(p_2)$, by IH, we have $p_1 \in F$ or $p_2 \in F$, which implies $p_1 \parallel_A p_2 \in F$.

(LTS2) It suffices to show that, for each p, if $p \notin F$ then $p \stackrel{\epsilon}{\Longrightarrow}_F |q$ for some q. Suppose $p \notin F$. By Lemma 4.1(7), there exists q such that $p \stackrel{\epsilon}{\Longrightarrow} |q$ and $q \notin F$. Then it immediately follows from Lemma 4.2 that $p \stackrel{\epsilon}{\Longrightarrow}_F |q$, as desired. □

A simple observation on proof trees for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \wedge qF$ is given below, which will be used in establishing a fundamental property of conjunction compositions.

Lemma 4.3. For any finite sequence $p_0 \wedge q_0 \stackrel{\tau}{\longrightarrow}, .., \stackrel{\tau}{\longrightarrow} p_i \wedge q_i \stackrel{\tau}{\longrightarrow}, .., \stackrel{\tau}{\longrightarrow} |p_n \wedge q_n (n \geq 0)$, if $p_i \wedge q_i \in F$ and $p_i, q_i \notin F$ for each $i \leq n$, then the inference of $p_0 \wedge q_0 F$ essentially depends on $p_n \wedge q_n F$, that is, each proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_0 \wedge q_0 F$ has a subtree with the root labelled with $p_n \wedge q_n F$, in particular, such subtree is proper if $n \geq 1$.

Proof. We prove it by induction on n. For the inductive basis $n = 0$, it holds trivially due to $p_0 \wedge q_0 \equiv p_n \wedge q_n$. For the inductive step, assume that $p_0 \wedge q_0 \stackrel{\tau}{\longrightarrow} p_1 \wedge q_1(\stackrel{\tau}{\longrightarrow})^k |p_{k+1} \wedge q_{k+1}$. Let $\mathcal{T}$ be any proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_0 \wedge q_0 F$. Since

$p_0, q_0 \notin F$ and $p_0 \wedge q_0 \xrightarrow{\tau}$, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{p_0 \wedge q_0 \xrightarrow{\alpha} r', \{rF : p_0 \wedge q_0 \xrightarrow{\alpha} r\}}{p_0 \wedge q_0 F} \text{ or } \frac{\{rF : p_0 \wedge q_0 \overset{\epsilon}{\Longrightarrow} |r\}}{p_0 \wedge q_0 F}.$$

For the first alternative, since $LTS(\text{CLL}_R)$ is τ-pure, we have $\alpha = \tau$. Then it follows from $p_0 \wedge q_0 \xrightarrow{\tau} p_1 \wedge q_1$ that, in the proof tree $\mathcal{T}$, one of nodes directly above the root is labelled with $p_1 \wedge q_1 F$. Thus, by IH, $\mathcal{T}$ has a proper subtree with the root labelled with $p_{k+1} \wedge q_{k+1} F$.

For the second alternative, since $p_0 \wedge q_0 \overset{\epsilon}{\Longrightarrow} |p_{k+1} \wedge q_{k+1}$, one of nodes directly above the root of $\mathcal{T}$ is labelled with $p_{k+1} \wedge q_{k+1} F$, as desired. $\Box$

The next three results has been obtained for CLL in pure process-algebraic style in (Zhang *et al.* 2011), where the proof essentially depends on the fact that, for any process p within CLL and $\alpha \in Act_\tau$, p is of more complex structure than its α-derivatives. Unfortunately, such property does not always hold for CLL_R. For instance, consider the process $\langle X | X = a.X \parallel_\emptyset a.b.X \rangle$. Here we give another proof along lines presented in (Zhu *et al.* 2013).

Lemma 4.4. If $p_1 \sqsubseteq_{RS} p_2$, $p_1 \sqsubseteq_{RS} p_3$ and $p_1 \notin F$ then $p_2 \wedge p_3 \notin F$.

Proof. Let $\Omega = \{q \wedge r : p \sqsubseteq_{RS} q, p \sqsubseteq_{RS} r \text{ and } p \notin F\}$. Clearly, it is enough to prove that $F \cap \Omega = \emptyset$. Conversely, suppose that $F \cap \Omega \neq \emptyset$. In the following, we intend to prove that, for each $t \in \Omega$, any proof tree of tF is not well-founded. Then a contradiction arises at this point due to Def. 2.4. Thus, to complete the proof, it suffices to show the claim below.

Claim For any $s \in \Omega$, each proof tree of sF has a proper subtree with the root labelled with $s'F$ for some $s' \in \Omega$.

Suppose $q \wedge r \in \Omega$. Then $p \sqsubseteq_{RS} q$, $p \sqsubseteq_{RS} r$ and $p \notin F$ for some p. Thus it follows that

$$q \notin F, r \notin F \text{ and } \mathcal{I}(p) = \mathcal{I}(q) = \mathcal{I}(r). \tag{4.4.1}$$

Let $\mathcal{T}$ be any proof tree of $Strip(\mathcal{P}_{\text{CLL}_R}, M_{\text{CLL}_R}) \vdash q \wedge rF$. By (4.4.1), the last rule applied in $\mathcal{T}$ is of the form

$$\text{either } \frac{\{sF : q \wedge r \overset{\epsilon}{\Longrightarrow} |s\}}{q \wedge rF} \text{ or } \frac{q \wedge r \xrightarrow{\alpha} s', \{sF : q \wedge r \xrightarrow{\alpha} s\}}{q \wedge rF}.$$

Since both q and r are stable, so is $q \wedge r$. Then, for the first alternative, the label of the node directly above the root of $\mathcal{T}$ is $q \wedge rF$ itself, as desired.

Next we consider the second alternative. In such case, $\tau \neq \alpha \in \mathcal{I}(q \wedge r)$ and

$$\forall s (q \wedge r \xrightarrow{\alpha} s \text{ implies } s \in F). \tag{4.4.2}$$

Hence $\alpha \in \mathcal{I}(q) \cap \mathcal{I}(r)$. Then $\alpha \in \mathcal{I}(p)$ due to (4.4.1). Further, since $p \notin F$, by Theorem 4.2, we get

$$p \xrightarrow{\alpha}_F p' \overset{\epsilon}{\Longrightarrow}_F |p'' \text{ for some } p' \text{ and } p''. \tag{4.4.3}$$

Then it immediately follows from $p \underset{\sim}{\sqsubset}_{RS} q$ and $p \underset{\sim}{\sqsubset}_{RS} r$ that

$$q \xrightarrow{\alpha}_F q' \overset{\epsilon}{\Longrightarrow}_F |q'' \text{ and } p'' \underset{\sim}{\sqsubset}_{RS} q'' \text{ for some } q', q'', \text{and} \tag{4.4.4}$$

$$r \xrightarrow{\alpha}_F r' \overset{\epsilon}{\Longrightarrow}_F |r'' \text{ and } p'' \underset{\sim}{\sqsubset}_{RS} r'' \text{ for some } r', r''. \tag{4.4.5}$$

So, $q \wedge r \xrightarrow{\alpha} q' \wedge r'$. Then $q' \wedge r' \in F$ by (4.4.2). Moreover, we obtain $q' \equiv q_0 \xrightarrow{\tau}_F, \ldots, \xrightarrow{\tau}_F |q_n \equiv q''$ for some q_i with $0 \leq i \leq n$, and $r' \equiv r_0 \xrightarrow{\tau}_F, \ldots, \xrightarrow{\tau}_F |r_m \equiv r''$ for some r_j with $0 \leq j \leq m$. Then

$$q' \wedge r' \equiv q_0 \wedge r_0 \xrightarrow{\tau}, .., \xrightarrow{\tau} q_n \wedge r_0 \xrightarrow{\tau} q_n \wedge r_1, .., \xrightarrow{\tau} |q_n \wedge r_m \equiv q'' \wedge r''. \tag{4.4.6}$$

By Lemma 4.2, it follows from $q' \wedge r' \in F$ that

$$q_i \wedge r_j \in F \text{ for each } q_i \wedge r_j \text{ occurring in (4.4.6)}. \tag{4.4.7}$$

It follows from (4.4.3), (4.4.4) and (4.4.5) that $q_n \wedge r_m \equiv q'' \wedge r'' \in \Omega$. Moreover, since one of nodes directly above the root of $\mathcal{T}$ is labelled with $q' \wedge r' F$, by (4.4.6), (4.4.7) and Lemma 4.3, it follows from $q_i \notin F$ and $r_j \notin F$ with $0 \leq i \leq n$ and $0 \leq j \leq m$ that $\mathcal{T}$ has a proper subtree with the root labelled with $q_n \wedge r_m F$. □

Lemma 4.5. If $p \underset{\sim}{\sqsubset}_{RS} q$ and $p \underset{\sim}{\sqsubset}_{RS} r$ then $p \underset{\sim}{\sqsubset}_{RS} q \wedge r$.

Proof. Set

$$\mathcal{R} = \{(p_1, p_2 \wedge p_3) : p_1 \underset{\sim}{\sqsubset}_{RS} p_2 \text{ and } p_1 \underset{\sim}{\sqsubset}_{RS} p_3\}.$$

It suffices to show that $\mathcal{R}$ is a stable ready simulation relation, which is almost immediate by using Lemma 4.4 to handle (RS2) and (RS3). □

We conclude this section with recalling a result obtained in (Lüttgen and Vogler 2010) and (Zhang *et al.* 2011) in different style, which reveals that $\sqsubseteq_{RS}$ is precongruent w.r.t the operators $\Box$, $\|_A$, $\vee$ and $\wedge$. Formally,

Theorem 4.3.

(1) For each $\odot \in \{\Box, \|_A, \wedge\}$, if $p \underset{\sim}{\sqsubset}_{RS} q$ and $s \underset{\sim}{\sqsubset}_{RS} r$ then $p \odot s \underset{\sim}{\sqsubset}_{RS} q \odot r$.

(2) For each $\odot \in \{\Box, \|_A, \vee, \wedge\}$, if $p \sqsubseteq_{RS} q$ and $s \sqsubseteq_{RS} r$ then $p \odot s \sqsubseteq_{RS} q \odot r$.

Proof. The item (2) follows from item (1). For item (1), the proofs are not much different from ones given in (Zhang *et al.* 2011). In particular, Lemma 4.5 is applied in the proof for the case $\odot = \wedge$. □

5. Basic properties of unfolding, context and transition

This section will provide a number of useful results that will be used in the following sections. In particular, we shall attend to elementary properties of unfolding and describing transitions in terms of contexts.

5.1. *Unfolding*

The notion of unfolding plays an important role in dealing with recursive operators. This subsection will give a few results concerning it. We begin with recalling the notion of unfolding.

Definition 5.1. Let X be a free variable in a given term t. An occurrence of X in t is said to be unfolded, if this occurrence does not occur in any scope of recursive operations $\langle Y|E\rangle$. Moreover, X is said to be unfolded if all occurrences of X in t are unfolded.

Definition 5.2 (Baeten and Bravetti 2008). A series of binary relations $\Rrightarrow_k$ over terms with $k < \omega$ is defined inductively as:

— $t \Rrightarrow_0 s$ if $t \equiv s$;
— $t \Rrightarrow_1 s$ if t has a subterm $\langle Y|E\rangle$ with $Y = t_Y \in E$ which is not in any scope of recursive operations, and s is obtained from t by replacing this subterm by $\langle t_Y|E\rangle$;
— $t \Rrightarrow_{k+1} s$ if $t \Rrightarrow_k t'$ and $t' \Rrightarrow_1 s$ for some term t'.

Moreover, $\Rrightarrow \triangleq \bigcup\limits_{0\le k<\omega} \Rrightarrow_k$. For any t and s, s is said to be a multi-step unfolding of t if $t \Rrightarrow s$.

For instance, consider $t \equiv (\langle X|X = a.X\Box b.\langle Y|Y = c.Y\rangle\rangle\Box d.0)\Box Z$, we have

$$t \Rrightarrow_1 ((a.\langle X|X = a.X\Box b.\langle Y|Y = c.Y\rangle\rangle\Box b.\langle Y|Y = c.Y\rangle)\Box d.0)\Box Z,$$

but it does not hold that $t \Rrightarrow_1 (\langle X|X = a.X\Box b.c.\langle Y|Y = c.Y\rangle\rangle\Box d.0)\Box Z$ because the subterm $\langle Y|Y = c.Y\rangle$ is in the scope of the recursive operation $\langle X|X = a.X\Box b.\langle Y|Y = c.Y\rangle\rangle$. The simple result below provides an equivalent formulation of the binary relation $\Rrightarrow_1$.

Lemma 5.1. For any term t_1 and t_2, $t_1 \Rrightarrow_1 t_2$ iff there exists a term s and variable X such that

($1_\Rrightarrow$) X is a unfolded variable in s,
($2_\Rrightarrow$) X occurs in s exactly once, and
($3_\Rrightarrow$) $t_1 \equiv s\{\langle Y|E\rangle/X\}$ and $t_2 \equiv s\{\langle t_Y|E\rangle/X\}$ for some Y, E with $Y = t_Y \in E$.

Proof. Immediately follows from Def. 5.2. □

A few trivial but useful properties concerning $\Rrightarrow_n$ are listed in the next lemma.

Lemma 5.2. For any term t, s and $X \in FV(t)$, if $t \Rrightarrow_n s$ then

(1) if X is unfolded in t then so is it in s and the number of occurrences of X in s is equal to that in t;
(2) the number of unguarded occurrences of X in s is not more than that in t;
(3) if X is (strongly) guarded in t then so is it in s;
(4) $FV(s) \subseteq FV(t)$;
(5) if X occurs in the scope of conjunction in s (that is, there exists a subterm $t_1 \wedge t_2$ of s such that X occurs in either t_1 or t_2) then so does it in t.

Proof. By Lemma 5.1 and Convention 3.2, it is straightforward by induction on n. □

Notice that the clause (2) in the above lemma does not always hold for guarded occurrences. For example, consider $t \equiv \langle X|X = a.X \wedge b.Y\rangle$, we have $t \Rrightarrow_1 a.\langle X|X = a.X \wedge b.Y\rangle \wedge b.Y$, and Y guardedly occurs in the latter twice but occurs in t only once. Clearly, the clause (2) strongly depends on Convention 3.2. Moreover, the clause (4) cannot be strengthened to "$FV(s) = FV(t)$". Consider $t \equiv \langle X_1|\{X_1 = a.0, X_2 = b.X_1 \Box Y\}\rangle$ and $t \Rrightarrow_1 a.0$, then we have $FV(t) = \{Y\}$ and $FV(a.0) = \emptyset$.

Given a variable X and term t, the folding number of X in t, in symbols $FN(t, X)$, is defined as the sum of depths of nesting recursive operations surrounding all *unguarded* occurrences of X in t. Formally,

Definition 5.3 (Folding number). Given a term t and $X \in FV(t)$, the folding number of X in t, denoted by $FN(t, X)$, is defined recursively below, where $UFV(t)$ is the set of all free variables which have unguarded occurrence in t.

— $FN(0, X) = FN(\bot, X) = FN(Y, X) = FN(X, X) = FN(t_1 \vee t_2, X) = FN(\alpha.t, X) \triangleq 0$;

— $FN(t_1 \odot t_2, X) \triangleq FN(t_1, X) + FN(t_2, X)$ with $\odot \in \{\Box, \|_A, \wedge\}$;

— $$FN(\langle Y|E\rangle, X) \triangleq \begin{cases} 1 + \sum\limits_{Z = t_Z \in E} FN(t_Z, X), & \text{if } X \in UFV(\langle Y|E\rangle); \\ \\ 0, & \text{otherwise.} \end{cases}$$

For instance, consider $t \equiv \langle X|X = a.X \vee Y_1\rangle \Box \langle Z|Z = c.Z \Box Y_2\rangle$, then $FN(t, Y_1) = 0$ and $FN(t, Y_2) = 1$.

Lemma 5.3. For any term t, there exits a term s such that $t \Rrightarrow s$ and each unguarded occurrence of any free variable in s is unfolded.

Proof. It proceeds by induction on $n = \sum\limits_{X \in UFV(t)} FN(t, X)$. For the induction base $n = 0$, it is easy to see that for each $X \in FV(t)$, any unguarded occurrence of X in t must be unfolded. Thus t itself meets our requirement because of $t \Rrightarrow t$. For the inductive step $n = k+1$, due to $n = k+1 > 0$, t is of the format either $t_1 \odot t_2$ with $\odot \in \{\wedge, \|_A\, \Box\}$ or $\langle Y|E\rangle$. In the following, we shall proceed by induction on the structure of t. For the case $t \equiv t_1 \odot t_2$ with $\odot \in \{\wedge, \|_A, \Box\}$, it is straightforward by applying IH on t_1 and t_2. Next we consider the case $t \equiv \langle Y|E\rangle$ with $Y = t_Y \in E$.

Clearly, $UFV(\langle Y|E\rangle) \neq \emptyset$ because of $n > 0$. Since $\langle Y|E\rangle \Rrightarrow_1 \langle t_Y|E\rangle$, by Lemma 5.2(2)(4), we have

$$UFV(\langle t_Y|E\rangle) \subseteq UFV(\langle Y|E\rangle).$$

Moreover, by Convention 3.2 and the definition of $\langle t_Y|E\rangle$, it is not difficult to get $FN(\langle Y|E\rangle, X) > FN(\langle t_Y|E\rangle, X)$ for each $X \in UFV(\langle t_Y|E\rangle)$. Hence

$$\sum_{X \in UFV(\langle t_Y|E\rangle)} FN(\langle t_Y|E\rangle, X) < \sum_{X \in UFV(\langle Y|E\rangle)} FN(\langle Y|E\rangle, X).$$

Then, by IH on n, there exists s such that $\langle t_Y|E\rangle \Rrightarrow s$ and each unguarded occurrence of any free variable is unfolded in s. Moreover, $\langle Y|E\rangle \Rrightarrow s$ due to $\langle Y|E\rangle \Rrightarrow_1 \langle t_Y|E\rangle$. $\Box$

5.2. *Contexts and transitions*

Due to the rules Rp_{12}, Rp_{13} and Rp_{15}, in order to obtain further properties of the inconsistency predicate F, we often need to capture the connection between the formats of p and q for a given transition $p \xrightarrow{\alpha} q$. Clearly, if p involves recursive operations, q is not always a subterm of p and its format often depends on some unfolding of p. This subsection intends to explore this issue.

Definition 5.4 (Context). A context $C_{\widetilde{X}}$ is a term whose free variables are among $\widetilde{X} = \{X_1, ..., X_n\}$ $(n \geq 0)$. Given processes p_i with $i \leq n$, the term $C_{\widetilde{X}}\{p_1/X_1, ..., p_n/X_n\}$ ($C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, for short) is obtained from $C_{\widetilde{X}}$ by replacing X_i by p_i for each $i < n$ simultaneously. In particular, we use $C_{\widetilde{X}}\{p/\widetilde{X}\}$ to denote the result of replacing all variables in $\widetilde{X}$ by p. A context $C_{\widetilde{X}}$ is said to be stable if $C_{\widetilde{X}}\{0/\widetilde{X}\} \not\xrightarrow{\tau}$.

In the remainder of this paper, whenever the expression $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ occurs, we always assume that $|\widetilde{p}| = |\widetilde{X}|$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ is subject to Convention 3.1 (recursive variables occurring in $\widetilde{p}$ may be renamed if it is necessary), where $|\widetilde{X}|$ is the length of the tuple $\widetilde{X}$.

Definition 5.5 (Active). An occurrence of a free variable X in term t is said to be *active* if such occurrence is unguarded and unfolded. A free variable X in term t is said to be active if its all occurrence are active. A free variable X in term t is said to be *1-active* if X occurs in t exactly once and such occurrence is active.

For example, X is 1-active in $\langle Y|Y = a.Y\rangle \Box X$. Moreover, it is evident that, for any context $C_{\widetilde{X}}$, if there exists an active occurrence of some variable within $C_{\widetilde{X}}$, then $C_{\widetilde{X}}$ is not of the form $\alpha.B_{\widetilde{X}}$, $B_{\widetilde{X}} \vee D_{\widetilde{X}}$ and $\langle Y|E\rangle$. This fact is used in demonstrating the next two lemmas, which give some properties of 1-active place-holder. Before presenting them, for simplicity of notation, we introduce the notation below.

Notation Given n-tuple processes $\widetilde{p} = \{p_1, \ldots, p_n\}$ and p', we use $\widetilde{p}[p'/p_i]$ to denote $\{p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_n\}$.

Lemma 5.4. For any $C_{\widetilde{X}}$ with 1-active variable X_{i_0} and $\widetilde{p}$ with $p_{i_0} \xrightarrow{\tau} p'$, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}}\{\widetilde{p}[p'/p_{i_0}]/\widetilde{X}\}$.

Proof. Proceeding by induction on the structure of $C_{\widetilde{X}}$. □

This result does not always hold for visible transitions. For instance, consider $C_X \equiv X\Box\tau.r$ and $p \equiv a.q$, although $p \xrightarrow{a} q$ and X is 1-active in C_X, it is false that $C_X\{p/X\} \xrightarrow{a}$.

Lemma 5.5. For any p and C_X with 1-active variable X, if $p \in F$ then $C_X\{p/X\} \in F$.

Proof. By a straightforward induction on C_X. □

In order to prove that $\sqsubseteq_{RS}$ still is precongruent in the presence of recursive operations, it is necessary to formally describe the contribution of $C_{\widetilde{X}}$ and $\widetilde{p}$ for a given transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$. In the following, we shall provide a few of results concerning this. We begin with considering τ-labelled transitions. Before giving the next lemma formally, we

illustrate the intuition behind it by means of an example. Consider $C_X \equiv (a.0 \vee X)\Box X$, $B_X \equiv \langle Y|Y = X\Box b.Y\rangle\Box c.0$, $p \equiv \tau.0$ and $q \equiv d.0$, then we have two τ-labelled transitions below

$$C_X\{q/X\} \xrightarrow{\tau} a.0\Box d.0$$

and

$$B_X\{p/X\} \xrightarrow{\tau} (0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle)\Box c.0.$$

It is not difficult to see that these two τ-labelled transitions depend on the capability of C_X and p respectively. In other words, for any q' and p' with $p' \xrightarrow{\tau}$, corresponding τ-transitions still exist for $C_X\{q'/X\}$ and $B_X\{p'/X\}$. Moreover, the targets have the same pattern. For instance, set $C'_X \equiv a.0\Box X$, then we have $C_X\{q/X\} \xrightarrow{\tau} C'_X\{q/X\}$ and $C_X\{q'/X\} \xrightarrow{\tau} C'_X\{q'/X\}$. We summarize this observation formally as follows, where two clauses capture τ-transitions exited by contexts and substitutions respectively, moreover, some simple properties on contexts are also listed in (C-τ-3) which will be used in the sequel.

Lemma 5.6. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$ then one of conclusions below holds.

(1) There exists $C'_{\widetilde{X}}$ such that

(C-τ-1) $r \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$;

(C-τ-2) for any processes $\widetilde{q}$, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$;

(C-τ-3) for each $X \in \widetilde{X}$,

(C-τ-3-i) if X is active in $C_{\widetilde{X}}$ then so is it in $C'_{\widetilde{X}}$ and the number of occurrences of X in $C'_{\widetilde{X}}$ is equal to that in $C_{\widetilde{X}}$;

(C-τ-3-ii) if X is unfolded in $C_{\widetilde{X}}$ then so is it in $C'_{\widetilde{X}}$ and the number of occurrences of X in $C'_{\widetilde{X}}$ is not more than that in $C_{\widetilde{X}}$;

(C-τ-3-iii) if X is strongly guarded in $C_{\widetilde{X}}$ then so is it in $C'_{\widetilde{X}}$;

(C-τ-3-iv) if X does not occur in any scope of conjunctions in $C_{\widetilde{X}}$ then nor does it in $C'_{\widetilde{X}}$.

(2) There exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i \leq |\widetilde{X}|$ such that

(P-τ-1) $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$, in particular, if X_i is active in $C_{\widetilde{X}}$ then $C'_{\widetilde{X}} \equiv C_{\widetilde{X}}$;

(P-τ-2) $p_i \xrightarrow{\tau} p'$ and $r \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}$ for some p';

(P-τ-3) $C''_{\widetilde{X},Z}\{X_i/Z\} \equiv C'_{\widetilde{X}}$ and Z is 1-active in $C''_{\widetilde{X},Z}$;

(P-τ-4) for any processes $\widetilde{q}$ with $q_i \xrightarrow{\tau} q'$, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}$.

Proof. It proceeds by induction on the depth of the inference of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$. We distinguish six cases based on the form of $C_{\widetilde{X}}$ as follows.

Case 1 $C_{\widetilde{X}}$ is closed.

Set $C'_{\widetilde{X}} \triangleq r$. Then (C-$\tau$-1,2,3) hold trivially.

Case 2 $C_{\widetilde{X}} \equiv X$ with $X \in \widetilde{X}$.

Put $C'_{\widetilde{X}} \triangleq X$ and $C''_{\widetilde{X},Z} \triangleq Z$ with $Z \notin \widetilde{X}$. Then it is easy to check that (P-τ-1) –

(P-τ-4) hold.

Case 3 $C_{\widetilde{X}} \equiv \alpha.B_{\widetilde{X}}$.

Thus $\alpha = \tau$ and $r \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$. Then it is not difficult to see that (C-τ-1,2,3) hold by taking $C'_{\widetilde{X}} \triangleq B_{\widetilde{X}}$.

Case 4 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\widetilde{X}}$.

Obviously, $r \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ or $r \equiv D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$. W.l.o.g, assume that $r \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$. We set $C'_{\widetilde{X}} \triangleq B_{\widetilde{X}}$. Then it is straightforward that (C-τ-1,2) and (C-τ-3-ii,iii,iv) hold. Moreover, since $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\widetilde{X}}$, for each $X \in \widetilde{X}$, each occurrence of X is weakly guarded. Hence (C-τ-3-i) holds trivially.

Case 5 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \odot D_{\widetilde{X}}$ with $\odot \in \{\Box, \wedge, \|_A\}$.

We consider the case $\odot = \Box$, others may be handled similarly and omitted. W.l.o.g, assume the last rule applied in the inference is

$$\frac{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r'}{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\Box D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r'\Box D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}}.$$

Then $r \equiv r'\Box D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$. For the τ-labelled transition $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r'$, by IH, either the clause (1) or (2) holds.

For the former case, there exists $B'_{\widetilde{X}}$ that satisfies (C-τ-1,2,3). Put $C'_{\widetilde{X}} \triangleq B'_{\widetilde{X}}\Box D_{\widetilde{X}}$. It immediately follows that $C'_{\widetilde{X}}$ satisfies (C-τ-1,2,3).

Next we consider the later case. In such situation, there exist $B'_{\widetilde{X}}$, $B''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i_0 \leq |\widetilde{X}|$ that satisfy (P-τ-1) – (P-τ-4). Set

$$C'_{\widetilde{X}} \triangleq B'_{\widetilde{X}}\Box D_{\widetilde{X}} \text{ and } C''_{\widetilde{X},Z} \triangleq B''_{\widetilde{X},Z}\Box D_{\widetilde{X}}.$$

We shall show that, for the τ-labelled transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$, $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ and i_0 realize (P-τ-1) – (P-τ-4).

(P-τ-1) It follows from $B_{\widetilde{X}} \Rrightarrow B'_{\widetilde{X}}$ that $C_{\widetilde{X}} \equiv B_{\widetilde{X}}\Box D_{\widetilde{X}} \Rrightarrow B'_{\widetilde{X}}\Box D_{\widetilde{X}} \equiv C'_{\widetilde{X}}$. If X_{i_0} is active in $C_{\widetilde{X}}$ then so is it in $B_{\widetilde{X}}$, and hence $C'_{\widetilde{X}} \equiv C_{\widetilde{X}}$ due to $B'_{\widetilde{X}} \equiv B_{\widetilde{X}}$.

(P-τ-2) Since $B'_{\widetilde{X}}$ satisfies (P-τ-2), $p_{i_0} \xrightarrow{\tau} p'$ and $r' \equiv B''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}$ for some p'. Due to $Z \notin \widetilde{X}$, we have $r \equiv B''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}\Box D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}$.

(P-τ-3) It follows from $B''_{\widetilde{X},Z}\{X_{i_0}/Z\} \equiv B'_{\widetilde{X}}$ and $Z \notin \widetilde{X}$ that $C''_{\widetilde{X},Z}\{X_{i_0}/Z\} \equiv B''_{\widetilde{X},Z}\{X_{i_0}/Z\}\Box D_{\widetilde{X}} \equiv C'_{\widetilde{X}}$. Moreover, since Z is 1-active in $B''_{\widetilde{X},Z}$, so is it in $C''_{\widetilde{X},Z}$.

(P-τ-4) Let $\widetilde{q}$ be any tuple with $|\widetilde{q}| = |\widetilde{p}|$ and $q_{i_0} \xrightarrow{\tau} q'$. It follows from $B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} B''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}$ and $Z \notin \widetilde{X}$ that $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}$.

Case 6 $C_{\widetilde{X}} \equiv \langle Y|E\rangle$.

Clearly, the last rule applied in the inference is

$$\frac{\langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r}{\langle Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r} \text{ with } Y = t_Y \in E.$$

For the τ-labelled transition $\langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$, by IH, either the clause (1) or (2) holds.

For the first alternative, there exists $C'_{\widetilde{X}}$ satisfying (C-τ-1,2,3). Then it is not difficult to check that, for the transition $\langle Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$, $C'_{\widetilde{X}}$ also realizes the conditions (C-τ-1,2,3). Here $\langle Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \Rrightarrow_1 \langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\}$ and Lemma 5.2(3)(5) are used to assert (C-τ-3-iii,iv) to be true.

For the second alternative, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i_0 \leq |\widetilde{X}|$ that satisfy (P-τ-1,2,3,4). Clearly, $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ and i_0 also realize (P-τ-1,2,3,4) for the transition $\langle Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$. In particular, $\langle Y|E\rangle \Rrightarrow C'_{\widetilde{X}}$ follows from $\langle Y|E\rangle \Rrightarrow_1 \langle t_Y|E\rangle \Rrightarrow C'_{\widetilde{X}}$. □

As an immediate consequence of Lemma 5.6, we have

Lemma 5.7. For any context $C_{\widetilde{X}}$, $C_{\widetilde{X}}$ is stable iff $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}$ for some $\widetilde{p}$.

Proof. Straightforward by Lemma 5.6. □

The next lemma is an instance of Lemma 5.6, which considers the case where the substitution is of the form $\langle Y|E\rangle$ or $\langle t_Y|E\rangle$.

Lemma 5.8. For any Y, E with $Y = t_Y \in E$ and context C_X with at most one occurrence of the unfolded variable X, we have

(1) if $C_X\{\langle Y|E\rangle/X\} \xrightarrow{\tau} q$ then there exists B_X such that

(1.1) $q \equiv B_X\{\langle Y|E\rangle/X\}$,
(1.2) $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{\tau} B_X\{\langle t_Y|E\rangle/X\}$, and
(1.3) X occurs in B_X at most once, moreover, such occurrence is unfolded;

(2) if $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{\tau} q$ then there exists B_X such that

(2.1) $q \equiv B_X\{\langle t_Y|E\rangle/X\}$,
(2.2) $C_X\{\langle Y|E\rangle/X\} \xrightarrow{\tau} B_X\{\langle Y|E\rangle/X\}$, and
(2.3) X occurs in B_X at most once, moreover, such occurrence is unfolded.

Proof. We prove only item (1), and the same reasoning applies to item (2). Assume $C_X\{\langle Y|E\rangle/X\} \xrightarrow{\tau} q$. Then, for such transition, by Lemma 5.6, either there exists C'_X that satisfies (C-τ-1,2,3) or there exist C'_X, $C''_{X,Z}$ with $Z \neq X$ that satisfy (P-τ-1,2,3,4).

For the first alternative, it follows from C'_X satisfies (C-τ-1,2) that $q \equiv C'_X\{\langle Y|E\rangle/X\}$ and $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{\tau} C'_X\{\langle t_Y|E\rangle/X\}$. Moreover, due to (C-$\tau$-3-ii), there is at most one occurrence of the unfolded variable X in C'_X. Consequently, the context C'_X is exactly one that we seek.

For the second alternative, by (P-τ-2), there exists q' such that

$$\langle Y|E\rangle \xrightarrow{\tau} q' \text{ and } q \equiv C''_{X,Z}\{\langle Y|E\rangle/X, q'/Z\}.$$

Hence $\langle t_Y|E\rangle \xrightarrow{\tau} q'$. Then it follows from (P-τ-4) that

$$C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{\tau} C''_{X,Z}\{\langle t_Y|E\rangle/X, q'/Z\}.$$

On the other hand, due to (P-τ-1), by Lemma 5.2(1), there is at most one occurrence of the unfolded variable X in C'_X. Moreover, since C'_X and $C''_{X,Z}$ satisfy (P-τ-3), we obtain $X \notin FV(C''_{X,Z})$. Hence $q \equiv C''_{X,Z}\{\langle Y|E\rangle/X, q'/Z\} \equiv C''_{X,Z}\{\langle t_Y|E\rangle/X, q'/Z\}$. Then it is easy to see that $B_X \triangleq q$ is what we need. □

In the following, we intend to provide an analogue of Lemma 5.6 for transitions labelled with visible actions. To explain intuition behind the next result clearly, it is best to work with an example. Consider $C_{X_1,X_2} \equiv ((X_1 \wedge \langle Y|Y = a.Y\rangle)\Box a.b.0) \parallel_{\{b\}} (X_1 \wedge X_2)$, $p_1 \equiv a.0$ and $p_2 \equiv a.c.0$, we have three a-labelled transitions below

$$C_{X_1,X_2}\{p_1/X_1, p_2/X_2\} \xrightarrow{a} (0 \wedge \langle Y|Y = a.Y\rangle) \parallel_{\{b\}} (a.0 \wedge a.c.0),$$

$$C_{X_1,X_2}\{p_1/X_1, p_2/X_2\} \xrightarrow{a} b.0 \parallel_{\{b\}} (a.0 \wedge a.c.0),$$

and

$$C_{X_1,X_2}\{p_1/X_1, p_2/X_2\} \xrightarrow{a} ((a.0 \wedge \langle Y|Y = a.Y\rangle)\Box a.b.0) \parallel_{\{b\}} (0 \wedge c.0).$$

These visible transitions starting from $C_{X_1,X_2}\{p_1/X_1, p_2/X_2\}$ are activated by three distinct events. Clearly, both the context C_{X_1,X_2} and the substitution p_1 contribute to the first transition, while two latter transitions depend merely on the capability of C_{X_1,X_2} and $\widetilde{p_{1,2}}$ respectively. These three situations may be described uniformly in the lemma below. Here some additional properties on contexts are also listed in (CP-a-4), which will be useful in the sequel.

Lemma 5.9. For any $a \in Act$, $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$ then there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \emptyset$ satisfying the conditions below:

(CP-a-1) $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$;

(CP-a-2) for each $Y \in \widetilde{Y}$, Y is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$;

(CP-a-3) there exist $i_Y \leq |\widetilde{X}|$ with $Y \in \widetilde{Y}$ such that

(CP-a-3-i) $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{X_{i_Y}}/\widetilde{Y}\} \equiv C'_{\widetilde{X}}$;

(CP-a-3-ii) there exist p'_Y with $Y \in \widetilde{Y}$ such that $p_{i_Y} \xrightarrow{a} p'_Y$ and $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$;

(CP-a-3-iii) for any $\widetilde{q}$ with $|\widetilde{q}| = |\widetilde{X}|$ and $\widetilde{q'}$ such that $|\widetilde{q'}| = |\widetilde{Y}|$ and $q_{i_Y} \xrightarrow{a} q'_Y$ with $Y \in \widetilde{Y}$, if $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ is stable then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$;

(CP-a-4) for each $X \in \widetilde{X}$,

(CP-a-4-i) the number of occurrences of X in $C''_{\widetilde{X},\widetilde{Y}}$ is not more than that in $C'_{\widetilde{X},\widetilde{Y}}$;

(CP-a-4-ii) if X is active in $C'_{\widetilde{X},\widetilde{Y}}$ then so is it in $C''_{\widetilde{X},\widetilde{Y}}$;

(CP-a-4-iii) if X does not occur in any scope of conjunctions in $C_{\widetilde{X}}$ then nor does it in $C''_{\widetilde{X},\widetilde{Y}}$.

Proof. It proceeds by induction on the depth of the inference of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$. Due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \not\xrightarrow{\tau}$, it is impossible that $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\widetilde{X}}$. Thus we

can distinguish seven cases depending on the form of $C_{\widetilde{X}}$.

Case 1 $C_{\widetilde{X}}$ is closed.

Set $C'_{\widetilde{X}} \equiv C'_{\widetilde{X},\widetilde{Y}} \triangleq C_{\widetilde{X}}$ and $C''_{\widetilde{X},\widetilde{Y}} \triangleq r$ with $\widetilde{Y} = \emptyset$. Clearly, these contexts realize conditions (CP-a-i) ($1 \leq i \leq 4$) trivially.

Case 2 $C_{\widetilde{X}} \equiv X_{i_0}$ with $i_0 \leq |\widetilde{X}|$.

Put $C'_{\widetilde{X}} \triangleq X_{i_0}$ and $C'_{\widetilde{X},\widetilde{Y}} \equiv C''_{\widetilde{X},\widetilde{Y}} \triangleq Y$ with $Y \notin \widetilde{X}$. Then (CP-a-i) ($1 \leq i \leq 4$) follow immediately, in particular, for (CP-a-3), we take $i_Y \triangleq i_0$.

Case 3 $C_{\widetilde{X}} \equiv \alpha.B_{\widetilde{X}}$.

Then $\alpha = a$ and $r \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$. Put $C'_{\widetilde{X}} \equiv C'_{\widetilde{X},\widetilde{Y}} \triangleq \alpha.B_{\widetilde{X}}$ and $C''_{\widetilde{X},\widetilde{Y}} \triangleq B_{\widetilde{X}}$ with $\widetilde{Y} = \emptyset$. Obviously, these contexts are what we seek.

Case 4 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \Box D_{\widetilde{X}}$.

W.l.o.g, suppose that the last rule applied in the inference is

$$\frac{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r,\ D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}}{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Box D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r}.$$

By IH, for the a-labelled transition $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$, there exist $B'_{\widetilde{X}}$, $B'_{\widetilde{X},\widetilde{Y}}$ and $B''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \emptyset$ that satisfy (CP-a-1) – (CP-a-4). Set

$$C'_{\widetilde{X}} \triangleq B'_{\widetilde{X}} \Box D_{\widetilde{X}}, C'_{\widetilde{X},\widetilde{Y}} \triangleq B'_{\widetilde{X},\widetilde{Y}} \Box D_{\widetilde{X}} \text{ and } C''_{\widetilde{X},\widetilde{Y}} \triangleq B''_{\widetilde{X},\widetilde{Y}}.$$

Then it is not difficult to check that, for the a-labelled transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$, these contexts above realizes (CP-a-1) – (CP-a-4), as desired.

Case 5 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \wedge D_{\widetilde{X}}$.

In this situation, the last rule applied in the inference is

$$\frac{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r_1, D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r_2}{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r_1 \wedge r_2}$$

and $r \equiv r_1 \wedge r_2$. Then by IH, there exist $B'_{\widetilde{X}}$, $B'_{\widetilde{X},\widetilde{Y}}$ and $B''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \emptyset$ and, $D'_{\widetilde{X}}$, $D'_{\widetilde{X},\widetilde{Z}}$ and $D''_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{X} \cap \widetilde{Z} = \emptyset$ that realize (CP-a-1,2,3,4) for two a-labelled transitions involving in premises, respectively. W.l.o.g, we may assume $\widetilde{Y} \cap \widetilde{Z} = \emptyset$. Then it is straightforward to verify that, for the a-labelled transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$, the contexts $C'_{\widetilde{X}} \triangleq B'_{\widetilde{X}} \wedge D'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{V}} \triangleq B'_{\widetilde{X},\widetilde{Y}} \wedge D'_{\widetilde{X},\widetilde{Z}}$ and $C''_{\widetilde{X},\widetilde{V}} \triangleq B''_{\widetilde{X},\widetilde{Y}} \wedge D''_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{V} = \widetilde{Y} \cup \widetilde{Z}$ realize (CP-a-1) – (CP-a-4), as desired.

Case 6 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \parallel_A D_{\widetilde{X}}$.

Then the last rule applied in the proof tree is one of the following:

(6.1) $\frac{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}r_1,D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}r_2}{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\|_A D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}r_1\|_A r_2}$ with $a \in A$;

(6.2) $\frac{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}r',\ D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\not\xrightarrow{\tau}}{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\|_A D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}r'\|_A D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}}$ with $a \notin A$;

(6.3) $\frac{D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}r',\ B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\not\xrightarrow{\tau}}{B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\|_A D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\xrightarrow{a}B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\|_A r'}$ with $a \notin A$.

Among them, the argument for (6.1) is similar to one for Case 5. We shall consider the case (6.2), and (6.3) may be handled similarly. In such situation, $r \equiv r' \parallel_A D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$. Moreover, for the a-labelled transition $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r'$, by IH, there exist $B'_{\widetilde{X}}$, $B'_{\widetilde{X},\widetilde{Y}}$ and $B''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \emptyset$ that satisfy (CP-a-1) – (CP-a-4). Put

$$C'_{\widetilde{X}} \triangleq B'_{\widetilde{X}} \parallel_A D_{\widetilde{X}}, C'_{\widetilde{X},\widetilde{Y}} \triangleq B'_{\widetilde{X},\widetilde{Y}} \parallel_A D_{\widetilde{X}} \text{ and } C''_{\widetilde{X},\widetilde{Y}} \triangleq B''_{\widetilde{X},\widetilde{Y}} \parallel_A D_{\widetilde{X}}.$$

Next we want to show that these contexts realize (CP-a-1) – (CP-a-4).

(CP-a-1) It is obvious because of $B_{\widetilde{X}} \Rrightarrow B'_{\widetilde{X}}$.

(CP-a-2) For each $Y \in \widetilde{Y}$, since Y is 1-active in $B''_{\widetilde{X},\widetilde{Y}}$ and $B'_{\widetilde{X},\widetilde{Y}}$, so is it in $C''_{\widetilde{X},\widetilde{Y}}$ and $C'_{\widetilde{X},\widetilde{Y}}$ because of $\widetilde{X} \cap \widetilde{Y} = \emptyset$.

(CP-a-3) By IH, there exist $i_Y \leq |\widetilde{X}|$ with $Y \in \widetilde{Y}$ which realize subclauses (i)(ii)(iii) in (CP-a-3). In the following, we will verify that these i_Y also work well for the induction step. Clearly, it follows from $B'_{\widetilde{X},\widetilde{Y}}\{\widetilde{X_{i_Y}/Y}\} \equiv B'_{\widetilde{X}}$ and $\widetilde{X} \cap \widetilde{Y} = \emptyset$ that $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{X_{i_Y}/Y}\} \equiv C'_{\widetilde{X}}$. Hence these i_Y satisfy the subclause (CP-a-3-i) for the induction step. Moreover, due to $r' \equiv B''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y/Y}\}$ for some $p'_Y (Y \in \widetilde{Y})$ with $p_{i_Y} \xrightarrow{a} p'_Y$ and $\widetilde{X} \cap \widetilde{Y} = \emptyset$, we have $r \equiv r' \parallel_A D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y/Y}\}$, that is, they realize (CP-a-3-ii) for the induction step. Finally, to verify that these i_Y also meet the challenge of (CP-a-3-iii), we assume that $\widetilde{q}$ and $\widetilde{q'}$ be any tuple such that $|\widetilde{q}| = |\widetilde{X}|$, $q_{i_Y} \xrightarrow{a} q'_Y$ for each $Y \in \widetilde{Y}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ is stable. So, $B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ and $D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ are stable. Further, since $B''_{\widetilde{X},\widetilde{Y}}$ satisfies (CP-a-3-iii) and $a \notin A$, it is easy to obtain that $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y/Y}\}$.

(CP-a-4) All subclauses immediately follow from IH and constructions of $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$.

Case 7 $C_{\widetilde{X}} \equiv \langle Y|E\rangle$.

Clearly, the last rule applied in the inference is

$$\frac{\langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r}{\langle Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r} \text{ with } Y = t_Y \in E.$$

For the transition $\langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$, by IH, there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \emptyset$ that satisfy (CP-a-1) – (CP-a-4). It is trivial to check that these contexts are what we need. □

Clearly, whenever all free variables occurring in $C_{\widetilde{X}}$ are guarded, any action labelled transition starting from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ must be excitated by $C_{\widetilde{X}}$ itself.

Lemma 5.10. Let $C_{\widetilde{X}}$ be a context such that X is guarded for each $X \in \widetilde{X}$. If $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\alpha} r$ then there exists $B_{\widetilde{X}}$ such that $r \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\alpha} B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any $\widetilde{q}$.

Proof. Firstly, we handle the case $\alpha = \tau$. For the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$, either the clause (1) or (2) in Lemma 5.6 holds. It is a simple matter to see that the clause (1) implies what we desire. The task is now to show that the clause (2) does not hold for such transition. On the contrary, assume that the clause (2) holds. Then there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ and $i_0 \leq |\widetilde{X}|$ satisfying (P-τ-1,2,3,4). For each $X \in \widetilde{X}$, since it is guarded in $C_{\widetilde{X}}$, by Lemma 5.2(3) and (P-τ-1), so is it in $C'_{\widetilde{X}}$. Hence a contradiction arises due to (P-τ-3), as desired.

Next we treat another case $\alpha \in \mathit{Act}$. By Lemma 5.9, for the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\alpha} r$, there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ realizing (CP-a-1) – (CP-a-4). Clearly, if $\widetilde{Y} = \emptyset$ then $C''_{\widetilde{X},\widetilde{Y}}$ is exactly one that we need. Thus, to complete the proof, it suffices to show that $\widetilde{Y}$ indeed is empty. Since X is guarded in $C_{\widetilde{X}}$ for each $X \in \widetilde{X}$ and $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$ (i.e., (CP-a-1)), by Lemma 5.2(3), all occurrences of free variables in $C'_{\widetilde{X}}$ are guarded. Moreover, since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (CP-a-2) and (CP-a-3-i), we get $\widetilde{Y} = \emptyset$, as desired. □

Analogous to Lemma 5.6 and 5.8, a particular instance of Lemma 5.9 is given as

Lemma 5.11. For any Y, E with $Y = t_Y \in E$ and context C_X with at most one occurrence of the unfolded variable X, we have
(1) if $C_X\{\langle Y|E\rangle/X\} \xrightarrow{a} q$ then there exists B_X such that
(1.1) $q \equiv B_X\{\langle Y|E\rangle/X\}$,
(1.2) $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{a} B_X\{\langle t_Y|E\rangle/X\}$, and
(1.3) X occurs in B_X at most once, moreover, such occurrence is unfolded;
(2) if $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{a} q$ then there exists B_X such that
(2.1) $q \equiv B_X\{\langle t_Y|E\rangle/X\}$,
(2.2) $C_X\{\langle Y|E\rangle/X\} \xrightarrow{a} B_X\{\langle Y|E\rangle/X\}$, and
(2.3) X occurs in B_X at most once, moreover, such occurrence is unfolded.

Proof. We handle only item (1), similar arguments apply to (2). Let $C_X\{\langle Y|E\rangle/X\} \xrightarrow{a} q$. Then, by Lemma 5.9, there exist C'_X, $C'_{X,\widetilde{Z}}$ and $C''_{X,\widetilde{Z}}$ with $X \notin \widetilde{Z}$ that satisfy (CP-a-1) – (CP-a-4). Since $C_X\{\langle Y|E\rangle/X\}$ is stable, by Lemma 5.8, so is $C_X\{\langle t_Y|E\rangle/X\}$.

If $\widetilde{Z} = \emptyset$, it follows trivially by (CP-a-3-iii) that $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{a} C''_{X,\widetilde{Z}}\{\langle t_Y|E\rangle/X\}$, moreover, by (CP-$a$-1), (CP-$a$-3-i), (CP-$a$-4-i) and Lemma 5.2(1), there is at most one occurrence of the unfolded variable X in $C''_{X,\widetilde{Z}}$. Therefore, $C''_{X,\widetilde{Z}}$ is exactly the context that we need.

We next deal with another case $\widetilde{Z} \neq \emptyset$. Since C'_X satisfies (CP-a-1), by Lemma 5.2(1), there is at most one occurrence of the unfolded variable X in C'_X. Then it follows from (CP-a-2), (CP-a-3-i) and (CP-a-4-i) that $|\widetilde{Z}| = 1$ and $X \notin FV(C''_{X,\widetilde{Z}})$. So, due to (CP-$a$-3-ii), there exists q' such that

$$\langle Y|E\rangle \xrightarrow{a} q' \text{ and } q \equiv C''_{X,\widetilde{Z}}\{\langle Y|E\rangle/X, q'/\widetilde{Z}\}.$$

Hence $\langle t_Y|E\rangle \xrightarrow{a} q'$. Then $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{a} C''_{X,\widetilde{Z}}\{\langle t_Y|E\rangle/X, q'/\widetilde{Z}\}$ by (CP-a-3-iii) and $C_X\{\langle t_Y|E\rangle/X\} \not\xrightarrow{\tau}$. Thus $q \equiv C''_{X,\widetilde{Z}}\{\langle t_Y|E\rangle/X, q'/\widetilde{Z}\}$ because of $X \notin FV(C''_{X,\widetilde{Z}})$. Then it is easy to check that $B_X \triangleq q$ is exactly what we seek. □

5.3. *More on unfolding*

Based on the result obtained in the preceding subsections, we shall give a few further properties of unfolding. We first want to indicate some simple properties.

Lemma 5.12. The relation $\Rrightarrow$ satisfies the forward and backward conditions, that is, for any $\alpha \in Act_\tau$ and p, q such that $p \Rrightarrow q$, we have

(1) if $p \xrightarrow{\alpha} p'$ then $q \xrightarrow{\alpha} q'$ and $p' \Rrightarrow q'$ for some q';
(2) if $q \xrightarrow{\alpha} q'$ then $p \xrightarrow{\alpha} p'$ and $p' \Rrightarrow q'$ for some p'.

Proof. **(1)** Assume $p \Rrightarrow q$ and $p \xrightarrow{\alpha} p'$. Clearly, $p \Rrightarrow_n q$ for some n. It proceeds by induction on n. For the induction base $n = 0$, it holds trivially. For the induction step $n = k + 1$, we have $p \Rrightarrow_k r \Rrightarrow_1 q$ for some r. By IH, $r \xrightarrow{\alpha} r'$ and $p' \Rrightarrow r'$ for some r'. Moreover, for $r \Rrightarrow_1 q$, by Lemma 5.1, 5.8(1) and 5.11(1), there exists q' such that $q \xrightarrow{\alpha} q'$ and $r' \Rrightarrow q'$. Obviously, we also have $p' \Rrightarrow q'$.

(2) Similar to item (1), but applying Lemma 5.8(2) and 5.11(2) instead of Lemma 5.8(1) and 5.11(1). □

Similar result also hold w.r.t $\overset{\epsilon}{\Longrightarrow}|$, that is

Lemma 5.13. For any p, q such that $p \Rrightarrow q$, we have

(1) if $p \overset{\epsilon}{\Longrightarrow} |p'$ then $q \overset{\epsilon}{\Longrightarrow} |q'$ and $p' \Rrightarrow q'$ for some q';
(2) if $q \overset{\epsilon}{\Longrightarrow} |q'$ then $p \overset{\epsilon}{\Longrightarrow} |q'$ and $p' \Rrightarrow q'$ for some p'.

Proof. By applying Lemma 5.12 finitely times. □

In fact, for any p, q such that $p \Rrightarrow q$, it is to be expected that $p =_{RS} q$. To verify it, we need to prove that $p \in F$ if and only if $q \in F$. The next lemma will serve as a stepping stone in proving this.

Convention 5.1. The arguments in the remainder of this paper often proceed by distinguishing some cases based on the last rule applied in an inference. For such argument, since rules about operations $\wedge$, $\vee$, $\|_A$ and $\Box$ are symmetric w.r.t their two operands (for instance, Rp_{11} and Rp_{12}, Ra_4 and Ra_5, and so on), we shall consider only one of two symmetric rules and omit another one.

Lemma 5.14. For any Y, E with $Y = t_Y \in E$ and context C_X with at most one occurrence of the unfolded variable X, $C_X\{\langle Y|E\rangle/X\} \in F$ iff $C_X\{\langle t_Y|E\rangle/X\} \in F$.

Proof. We give proof only for the implication from left to right, the converse implication may be proved similarly and omitted. Assume $C_X\{\langle Y|E\rangle/X\} \in F$. It proceeds by induction on the depth of the inference $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{\langle Y|E\rangle/X\}F$,

which is a routine case analysis on the form of C_X. We give the proof only for the case $C_X \equiv B_X \wedge D_X$, the other cases are left to the reader. In this situation, the last rule applied in the inference is one of the following.

Case 1 $\frac{B_X\{\langle Y|E\rangle/X\}F}{B_X\{\langle Y|E\rangle/X\}\wedge D_X\{\langle Y|E\rangle/X\}F}$.
By IH, we get $B_X\{\langle t_Y|E\rangle/X\} \in F$. Hence $C_X\{\langle t_Y|E\rangle/X\} \in F$.

Case 2 $\frac{B_X\{\langle Y|E\rangle/X\}\stackrel{a}{\longrightarrow}r, D_X\{\langle Y|E\rangle/X\}\stackrel{a}{\not\longrightarrow}, C_X\{\langle Y|E\rangle/X\}\stackrel{\tau}{\not\longrightarrow}}{B_X\{\langle Y|E\rangle/X\}\wedge D_X\{\langle Y|E\rangle/X\}F}$.
By Lemma 5.8 and 5.11, we have $B_X\{\langle t_Y|E\rangle/X\} \stackrel{a}{\longrightarrow}$, $D_X\{\langle t_Y|E\rangle/X\} \stackrel{a}{\not\longrightarrow}$ and $C_X\{\langle t_Y|E\rangle/X\} \stackrel{\tau}{\not\longrightarrow}$. So, $C_X\{\langle t_Y|E\rangle/X\} \in F$.

Case 3 $\frac{C_X\{\langle Y|E\rangle/X\}\stackrel{\alpha}{\longrightarrow}s, \{rF: C_X\{\langle Y|E\rangle/X\}\stackrel{\alpha}{\longrightarrow}r\}}{C_X\{\langle Y|E\rangle/X\}F}$.
Then $C_X\{\langle t_Y|E\rangle/X\} \stackrel{\alpha}{\longrightarrow}$ by Lemma 5.8(1) and 5.11(1). Assume $C_X\{\langle t_Y|E\rangle/X\} \stackrel{\alpha}{\longrightarrow} q$. Thus it follows from Lemma 5.8(2) and 5.11(2) that there exists C'_X with at most one occurrence of the unfolded variable X such that

$$C_X\{\langle Y|E\rangle/X\} \stackrel{\alpha}{\longrightarrow} C'_X\{\langle Y|E\rangle/X\} \text{ and } q \equiv C'_X\{\langle t_Y|E\rangle/X\}.$$

Then, by IH, $q \equiv C'_X\{\langle t_Y|E\rangle/X\} \in F$. Hence $C_X\{\langle t_Y|E\rangle/X\} \in F$ by Theorem 4.2.

Case 4 $\frac{\{rF: C_X\{\langle Y|E\rangle/X\}\stackrel{\epsilon}{\Longrightarrow}|r\}}{C_X\{\langle Y|E\rangle/X\}F}$.
Assume $C_X\{\langle t_Y|E\rangle/X\} \stackrel{\epsilon}{\Longrightarrow} |t$. Repeated application of Lemma 5.8(2) enables us to get $C_X\{\langle Y|E\rangle/X\} \stackrel{\epsilon}{\Longrightarrow} |r$, $r \equiv C'_X\{\langle Y|E\rangle/X\}$ and $t \equiv C'_X\{\langle t_Y|E\rangle/X\}$ for some r and context C'_X with at most one occurrence of the unfolded free variable X. Since $r \equiv C'_X\{\langle Y|E\rangle/X\} \in F$, we have $t \in F$ by IH. Then $C_X\{\langle t_Y|E\rangle/X\} \in F$ by Theorem 4.2. □

Next we can show that the relation $\Rrightarrow$ preserves and respects the inconsistency.

Lemma 5.15. For any p, q, if $p \Rrightarrow q$, then $p \in F$ iff $q \in F$.

Proof. Suppose $p \Rrightarrow q$. Hence $p \Rrightarrow_n q$ for some n. Then, using Lemma 5.1 and 5.14, the proof is straightforward by induction on n. □

We now have the below assertion of the equivalence of p and q modulo $=_{RS}$ whenever $p \Rrightarrow q$.

Lemma 5.16. If $p_1 \Rrightarrow p_2$ then $p_1 =_{RS} p_2$, in particular, $p_1 \approx_{RS} p_2$ whenever $p_1 \stackrel{\tau}{\not\longrightarrow}$.

Proof. We only prove $p_1 \sqsubset_{\sim RS} p_2$ whenever $p_1 \stackrel{\tau}{\not\longrightarrow}$, other proofs are straightforward and omitted. Set

$$\mathcal{R} = \{(p, q) : p \Rrightarrow q \text{ and } p \stackrel{\tau}{\not\longrightarrow}\}.$$

It suffices to prove that $\mathcal{R}$ is a stable ready simulation relation. Suppose $(p, q) \in \mathcal{R}$. Then, by Lemma 5.12 and 5.15, it is evident that such pair satisfies (RS1), (RS2) and (RS4). For (RS3), suppose $p \stackrel{a}{\Longrightarrow}_F |p'$. Then $p \stackrel{a}{\longrightarrow}_F p'' \stackrel{\epsilon}{\Longrightarrow}_F |p'$ for some p''. By Lemma 5.12

and 5.15, there exists q'' such that $q \xrightarrow{a}_F q''$ and $p'' \Rrightarrow q''$. Further, by Lemma 5.13 and 5.15, $p' \Rrightarrow q'$ and $q'' \stackrel{\epsilon}{\Longrightarrow}_F |q'$ for some q'. Moreover, $(p', q') \in \mathcal{R}$, as desired. $\square$

In the following, we shall generalize Lemma 5.6 to the situation involving a sequence of τ-labelled transitions. Given a process $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, by Lemma 5.6, any τ-transition starting from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ may be caused by $C_{\widetilde{X}}$ itself or some p_i. Thus, for a sequence of τ-transitions, these two situations may occur alternately. Based on Lemma 5.6, we can capture this as follows.

Lemma 5.17. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} r$ then there exist $C'_{\widetilde{X},\widetilde{Y}}$, $i_Y \leq |\widetilde{X}|$ and p'_Y with $Y \in \widetilde{Y}$ such that

(MS-τ-1) $\widetilde{X} \cap \widetilde{Y} = \emptyset$ and Y is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ for each $Y \in \widetilde{Y}$;

(MS-τ-2) $p_{i_Y} \stackrel{\tau}{\Longrightarrow} p'_Y$ for each $Y \in \widetilde{Y}$ and $r \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$;

(MS-τ-3) for any $\widetilde{q}$ and $\widetilde{q'_Y}$ with $|\widetilde{q}| = |\widetilde{X}|$ and $Y \in \widetilde{Y}$,

  (MS-τ-3-i) if $q_{i_Y} \stackrel{\epsilon}{\Longrightarrow} q'_Y$ with $Y \in \widetilde{Y}$ then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}\Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$;

  (MS-τ-3-ii) if $q_{i_Y} \stackrel{\tau}{\Longrightarrow} q'_Y$ with $Y \in \widetilde{Y}$ then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$;

(MS-τ-4) if $C_{\widetilde{X}}$ is stable then so is $C'_{\widetilde{X},\widetilde{Y}}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q_{i_Y}}/\widetilde{Y}\}$ for any $\widetilde{q}$;

(MS-τ-5) for each $X \in \widetilde{X}$, if X is strongly guarded in $C_{\widetilde{X}}$ then so is it in $C'_{\widetilde{X},\widetilde{Y}}$ and $X \not\equiv X_{i_Y}$ for each $Y \in \widetilde{Y}$;

(MS-τ-6) for each $X \in \widetilde{X}$ (or, $Y \in \widetilde{Y}$), if X (X_{i_Y}, respectively) does not occur in any scope of conjunctions in $C_{\widetilde{X}}$ then nor does X (Y, respectively) in $C'_{\widetilde{X},\widetilde{Y}}$;

(MS-τ-7) if r is stable then so are $C'_{\widetilde{X},\widetilde{Y}}$ and p'_Y for each $Y \in \widetilde{Y}$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^n r (n \geq 0)$. We proceed by induction on n. For the inductive base $n = 0$, the conclusion holds trivially by taking $C'_{\widetilde{X},\widetilde{Y}} \triangleq C_{\widetilde{X}}$ with $\widetilde{Y} = \emptyset$.

For the inductive step, assume $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k s \stackrel{\tau}{\longrightarrow} r$ for some s. For the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k s$, by IH, there exist $C'_{\widetilde{X},\widetilde{Y}}$, $i_Y \leq |\widetilde{X}|$ and p'_Y ($Y \in \widetilde{Y}$) such that

$$C'_{\widetilde{X},\widetilde{Y}}, i_Y \text{ and } p'_Y \text{ with } Y \in \widetilde{Y} \text{ realize } (\mathrm{MS}-\tau-l)(1 \leq l \leq 7). \tag{5.17.1}$$

In particular, we have $s \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$ due to (MS-τ-2). Then, for the transition $s \stackrel{\tau}{\longrightarrow} r$, either the clause (1) or (2) in Lemma 5.6 holds. The argument splits into two cases.

Case 1 For the transition $s \stackrel{\tau}{\longrightarrow} r$, the clause (1) in Lemma 5.6 holds.

That is, for the transition $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \equiv s \stackrel{\tau}{\longrightarrow} r$, there exists $C''_{\widetilde{X},\widetilde{Y}}$ satisfying (C-τ-1,2,3) in Lemma 5.6. We shall check that $C''_{\widetilde{X},\widetilde{Y}}$, $\widetilde{i_Y}$ and $\widetilde{p'_Y}$ realize (MS-τ-1) - (MS-τ-7) w.r.t $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^{k+1} r$.

Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-1,5,6), it follows that $C''_{\widetilde{X},\widetilde{Y}}$ and $\widetilde{i_Y}$ realize (MS-τ-1), (MS-τ-5) and (MS-τ-6) due to (C-τ-3-i), (C-τ-3-iii) and (C-τ-3-iv) respectively. Moreover, as

$C''_{\widetilde{X},\widetilde{Y}}$ satisfies (C-τ-1) it follows immediately that (MS-τ-2) holds. Since $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (C-τ-2), by Lemma 5.7, $C'_{\widetilde{X},\widetilde{Y}}$ is not stable. Then nor is $C_{\widetilde{X}}$ because $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-4). Thus, $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-4) trivially.

Next we verify (MS-τ-3). Let $\widetilde{q}$ be any processes with $|\widetilde{q}| = |\widetilde{X}|$ and $q_{i_Y} \stackrel{\epsilon}{\Longrightarrow} q'_Y$ with $Y \in \widetilde{Y}$.

(MS-τ-3-i) Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-i), we have

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \text{ for some } t.$$

Moreover, we have $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \xrightarrow{\tau} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ due to (C-τ-2). Then it follows from Lemma 5.12 that

$$t \xrightarrow{\tau} t' \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \text{ for some } t'.$$

Therefore, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \xrightarrow{\tau} t' \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$, as desired.

(MS-τ-3-ii) It is straightforward as $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-ii) and $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (C-τ-2).

(MS-τ-7) Suppose $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \not\xrightarrow{\tau}$. Then, since $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-1), by Lemma 5.4 and 5.7, it is easy to see that both $C''_{\widetilde{X},\widetilde{Y}}$ and p'_Y with $Y \in \widetilde{Y}$ are stable.

Case 2 For the transition $s \xrightarrow{\tau} r$, the clause (2) in Lemma 5.6 holds.

Then there exist $i_0 \leq |\widetilde{X}| + |\widetilde{Y}|$, $C''_{\widetilde{X},\widetilde{Y}}(\equiv C''_{X_1,\dots,X_{|\widetilde{X}|},Y_{|\widetilde{X}|+1},\dots,Y_{|\widetilde{X}|+|\widetilde{Y}|}})$ and $C'''_{\widetilde{X},\widetilde{Y},Z}(\equiv C'''_{X_1,\dots,X_{|\widetilde{X}|},Y_{|\widetilde{X}|+1},\dots,Y_{|\widetilde{X}|+|\widetilde{Y}|},Z})$ with $Z \notin \widetilde{X} \cup \widetilde{Y}$ satisfying (P-τ-1) - (P-τ-4). In particular, by (P-τ-3),

$$C''_{\widetilde{X},\widetilde{Y}} \equiv \begin{cases} C'''_{\widetilde{X},\widetilde{Y},Z}\{X_{i_0}/Z\}, & \text{if } 1 \leq i_0 \leq |\widetilde{X}|; \\ \\ C'''_{\widetilde{X},\widetilde{Y},Z}\{Y_{i_0}/Z\}, & \text{if } |\widetilde{X}| + 1 \leq i_0 \leq |\widetilde{X}| + |\widetilde{Y}|. \end{cases}$$

For the case $|\widetilde{X}| + 1 \leq i_0 \leq |\widetilde{X}| + |\widetilde{Y}|$, by (P-$\tau$-2), there exists p' such that

$$p'_{Y_{i_0}} \xrightarrow{\tau} p' \text{ and } r \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}, p'/Z\}.$$

Moreover, since Y_{i_0} is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$, by (P-$\tau$-1), we have $C'_{\widetilde{X},\widetilde{Y}} \equiv C''_{\widetilde{X},\widetilde{Y}}$. Further, since Z is 1-active in $C'''_{\widetilde{X},\widetilde{Y},Z}$ and $C''_{\widetilde{X},\widetilde{Y}} \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{Y_{i_0}/Z\}$, it is easy to see that Y_{i_0} does not occur in $C'''_{\widetilde{X},\widetilde{Y},Z}$. Hence

$$r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}[p'/p'_{Y_{i_0}}]/\widetilde{Y}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}[p'/p'_{Y_{i_0}}]/\widetilde{Y}\}.$$

Then it is not difficult to check that $C'_{\widetilde{X},\widetilde{Y}}$, $\widetilde{p'_Y}[p'/p'_{Y_{i_0}}]$ and i_Y with $Y \in \widetilde{Y}$ realize (MS-τ-l)($1 \leq l \leq 7$) w.r.t the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\xrightarrow{\tau})^{k+1}r$, as desired.

For the case $1 \leq i_0 \leq |\widetilde{X}|$, by (P-$\tau$-2), $p_{i_0} \xrightarrow{\tau} p''$ for some p'' such that $r \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}, p''/Z\}$. Set

$$i_Z \triangleq i_0 \text{ and } p'_Z \triangleq p''.$$

In the following, we intend to verify that $C'''_{\widetilde{X},\widetilde{Y},Z}$, i_U ($U \in \widetilde{Y} \cup \{Z\}$) and $|\widetilde{Y}|+1$-tuple $\widetilde{p'_U}$ with $U \in \widetilde{Y} \cup \{Z\}$ realize (MS-τ-1) - (MS-τ-7) w.r.t $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^{k+1}r$.

(MS-τ-1) By (P-τ-1), we have $C'_{\widetilde{X},\widetilde{Y}} \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}$. Moreover, since $C'_{\widetilde{X},\widetilde{Y}}$ satisfy (MS-τ-1), by Lemma 5.2(1), Y is 1-active in $C''_{\widetilde{X},\widetilde{Y}}$ for each $Y \in \widetilde{Y}$. Further, by (P-τ-3), each $Y \in \widetilde{Y}$ and Z are 1-active in $C'''_{\widetilde{X},\widetilde{Y},Z}$.

(MS-τ-2) It is straightforward.

(MS-τ-3) Let $\widetilde{q}$ be any processes with $|\widetilde{q}| = |\widetilde{X}|$.

(MS-τ-3-i) Suppose $q_{i_U} \stackrel{\epsilon}{\Longrightarrow} q'_U$ with $U \in \widetilde{Y} \cup \{Z\}$. Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-i), we have

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \text{ for some } t.$$

It follows from $q_{i_Z} \stackrel{\epsilon}{\Longrightarrow} q'_Z$ that $q_{i_Z}(\stackrel{\tau}{\longrightarrow})^m q'_Z$ for some $m \geq 0$. We shall distinguish two cases based on m.

For the case $m = 0$, we get $q_{i_Z} \equiv q'_Z$. Since $C'''_{\widetilde{X},\widetilde{Y},Z}$ satisfies (P-τ-1) and (P-τ-3), we have

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}.$$

Therefore $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}$, as desired.

For the case $m > 0$, i.e., $q_{i_Z} \stackrel{\tau}{\longrightarrow} q'' \stackrel{\epsilon}{\Longrightarrow} q'_Z$ for some q'', by (P-τ-4), we obtain

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \stackrel{\tau}{\longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q''/Z\}.$$

Moreover, since Z is 1-active, by Lemma 5.4, we get

$$C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q''/Z\} \stackrel{\epsilon}{\Longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}.$$

Then, by Lemma 5.13, it follows from $t \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ that there exist t' such that

$$t \stackrel{\epsilon}{\Longrightarrow} t' \Rrightarrow C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}.$$

Consequently, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \stackrel{\epsilon}{\Longrightarrow} t' \Rrightarrow C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}$.

(MS-τ-3-ii) Suppose $q_{i_U} \stackrel{\tau}{\Longrightarrow} q'_U$ with $U \in \widetilde{Y} \cup \{Z\}$. Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-ii), we have

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}.$$

Moreover, $q_{i_Z} \stackrel{\tau}{\longrightarrow} q'' \stackrel{\epsilon}{\Longrightarrow} q'_Z$ for some q'' because of $q_{i_Z} \stackrel{\tau}{\Longrightarrow} q'_Z$. Hence by (P-$\tau$-4)

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \stackrel{\tau}{\longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q''/Z\}.$$

Further, since Z is 1-active, it follows from Lemma 5.4 that

$$C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q''/Z\} \stackrel{\epsilon}{\Longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}.$$

Consequently, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}, q'_Z/Z\}$, as desired.

(MS-τ-4) Assume $C_{\widetilde{X}}$ is stable. By (MS-τ-4), $C'_{\widetilde{X},\widetilde{Y}}$ is stable and for any $\widetilde{q}$, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \Rrightarrow$

$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q_{i_Y}}/\widetilde{Y}\}$. Moreover, by Lemma 5.12, it follows from $C'_{\widetilde{X},\widetilde{Y}} \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}$ (i.e., (P-τ-1)) and $C'''_{\widetilde{X},\widetilde{Y},Z}\{X_{i_Z}/Z\} \equiv C''_{\widetilde{X},\widetilde{Y}}$ (i.e., (P-τ-3)) that $C'''_{\widetilde{X},\widetilde{Y},Z}$ is stable and

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q_{i_Y}}/\widetilde{Y}\} \Rrightarrow C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q_{i_Y}}/\widetilde{Y},q_{i_Z}/Z\}.$$

(MS-τ-5,6) By Lemma 5.2(3)(5), they immediately follow from the fact that $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-5,6) and $C'''_{\widetilde{X},\widetilde{Y},Z}$ satisfies (P-τ-1,3).

(MS-τ-7) Immediately follows from (MS-τ-1), (MS-τ-2) and Lemma 5.4 and 5.7. □

Lemma 5.18. For any $\widetilde{p}$ and stable context $C_{\widetilde{X}}$, if, for each $i \leq |\widetilde{X}|$, $p_i \stackrel{\epsilon}{\Longrightarrow} |p'_i$ then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$ for some q.

Proof. By Lemma 5.3 and 5.2(4), $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$ for some $C'_{\widetilde{X}}$ such that each unguarded occurrence of any free variable in $C'_{\widetilde{X}}$ is unfolded. Moreover, since $C_{\widetilde{X}}$ is stable, so is $C'_{\widetilde{X}}$ by $C_{\widetilde{X}}\{0/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{0/\widetilde{X}\}$ and Lemma 5.12.

Let $C'_{\widetilde{X},\widetilde{Y}}$ be the context obtained from $C'_{\widetilde{X}}$ by replacing simultaneously all unguarded and unfolded occurrences of free variables in $\widetilde{X}$ by distinct and fresh variables $\widetilde{Y}$. Here distinct occurrences are replaced by distinct variables. Clearly, we have

(1) for each $Y \in \widetilde{Y}$, there exists exactly one $i_Y \leq |\widetilde{X}|$ such that $C'_{\widetilde{X}} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{X_{i_Y}}/\widetilde{Y}\}$,

(2) all variables in $\widetilde{Y}$ are 1-active in $C'_{\widetilde{X},\widetilde{Y}}$, and

(3) $C'_{\widetilde{X},\widetilde{Y}}$ is stable.

Then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p_{i_Y}}/\widetilde{Y}\}$, and by Lemma 5.4, we obtain $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p_{i_Y}}/\widetilde{Y}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p'_{i_Y}}/\widetilde{Y}\}$. Further, since $C'_{\widetilde{X},\widetilde{Y}}$ and $\widetilde{p'_{i_Y}}$ are stable and $\widetilde{Y}$ contains all unguarded occurrences of variables in $C'_{\widetilde{X},\widetilde{Y}}$, we get $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p'_{i_Y}}/\widetilde{Y}\} \stackrel{\tau}{\not\longrightarrow}$ by Lemma 5.6. Hence, by Lemma 5.13, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p'_{i_Y}}/\widetilde{Y}\}$ for some q. □

Given a process $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and its stable τ-descendant r (i.e., $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r$), in general there exist more than one evolution paths from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ to r. Since each τ-labelled transition in CLL_R activated by a single process, a natural conjecture arises at this point that there exist some "canonical" evolution paths from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ to r in which the context $C_{\widetilde{X}}$ itself evolves into a stable context then p_i evolves. A weak version of this conjecture will be verified in Lemma 5.20. To this end, a preliminary result is given below.

Lemma 5.19. Let t_1, t_2 be two terms and $\widetilde{X}$ a tuple of variables such that any recursive variable occurring in t_i(with $i = 1, 2$) is not in $\widetilde{X}$, and let $\widetilde{a_X.0}$ be a tuple of processes with fresh visible action a_X for each $X \in \widetilde{X}$. Then

(1) if $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$ then $t_1 \equiv t_2$;
(2) if $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow_1 t_2\{\widetilde{a_X.0}/\widetilde{X}\}$ then $t_1\{\widetilde{r}/\widetilde{X}\} \Rrightarrow_1 t_2\{\widetilde{r}/\widetilde{X}\}$ for any $\widetilde{r}$.

Proof. **(1)** If $FV(t_1) \cap \widetilde{X} = \emptyset$ then $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_1 \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$. Further, since a_X is fresh for each $X \in \widetilde{X}$, we have $FV(t_2) \cap \widetilde{X} = \emptyset$. Hence $t_1 \equiv t_2$. In the following, we

consider another case $FV(t_1) \cap \widetilde{X} \neq \emptyset$. We proceed by induction on t_1.

Case 1 $t_1 \equiv X_i$.
Then $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv a_{X_i}.0 \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$. Hence $t_2 \equiv X_i$ due to the freshness of a_{X_i}.

Case 2 $t_1 \equiv \alpha.s$.
So $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv \alpha.s\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$. Since $\alpha \neq a_X$ for each $X \in \widetilde{X}$, there exists s' such that $t_2 \equiv \alpha.s'$ and $s\{\widetilde{a_X.0}/\widetilde{X}\} \equiv s'\{\widetilde{a_X.0}/\widetilde{X}\}$. By IH, we have $s \equiv s'$. Hence $t_1 \equiv t_2$.

Case 3 $t_1 \equiv s_1 \odot s_2$ with $\odot \in \{\vee, \Box, \|_A, \wedge\}$.
Then $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv s_1\{\widetilde{a_X.0}/\widetilde{X}\} \odot s_2\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$. Since $\widetilde{a_X.0}$ do not contain $\odot$, there exist s_1', s_2' such that $t_2 \equiv s_1' \odot s_2'$, $s_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv s_1'\{\widetilde{a_X.0}/\widetilde{X}\}$ and $s_2\{\widetilde{a_X.0}/\widetilde{X}\} \equiv s_2'\{\widetilde{a_X.0}/\widetilde{X}\}$. Hence $s_1 \equiv s_1'$ and $s_2 \equiv s_2'$ by applying IH.

Case 4 $t_1 \equiv \langle Y|E\rangle$ for some $E(V)$ with $Y \in V$.
Then $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv \langle Y|E\{\widetilde{a_X.0}/\widetilde{X}\}\rangle \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$. So, $t_2 \equiv \langle Y|E'\rangle$ for some $E'(V)$ such that for each $Z \in V$, $t_Z\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_Z'\{\widetilde{a_X.0}/\widetilde{X}\}$ where $Z = t_Z \in E$ and $Z = t_Z' \in E'$. By IH, $t_Z \equiv t_Z'$ for each $Z \in V$. Thus $t_1 \equiv \langle Y|E\rangle \equiv \langle Y|E'\rangle \equiv t_2$.

(2) For the case $FV(t_1) \cap \widetilde{X} = \emptyset$, since one-step unfolding does not bring any fresh actions, we have $FV(t_2) \cap \widetilde{X} = \emptyset$. Thus, $t_1 \equiv t_1\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow_1 t_2\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_2$, and hence $t_1 \equiv t_1\{\widetilde{r}/\widetilde{X}\} \Rrightarrow_1 t_2\{\widetilde{r}/\widetilde{X}\} \equiv t_2$ for any $\widetilde{r}$. Next we consider another case $FV(t_1) \cap \widetilde{X} \neq \emptyset$. It proceeds by induction on t_1. This is a routine case analysis on the format of t_1, we handle only the case $t_1 \equiv \langle Y|E\rangle$.

In this case, $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv \langle Y|E\rangle\{\widetilde{a_X.0}/\widetilde{X}\}$. By Def. 5.2, the unique result of one-step unfolding of $\langle Y|E\rangle\{\widetilde{a_X.0}/\widetilde{X}\}$ is $\langle t_Y|E\rangle\{\widetilde{a_X.0}/\widetilde{X}\}$ where $Y = t_Y \in E$. Thus, we get $\langle t_Y|E\rangle\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$. By item (1), we have $\langle t_Y|E\rangle \equiv t_2$, and hence $t_1\{\widetilde{r}/\widetilde{X}\} \Rrightarrow_1 \langle t_Y|E\rangle\{\widetilde{r}/\widetilde{X}\} \equiv t_2\{\widetilde{r}/\widetilde{X}\}$ for any $\widetilde{r}$. $\Box$

Having disposed of this preliminary step, we can now verify a weak version of the conjecture above mentioned, which is enough for the aim of this paper. At present, we do not know whether this result still holds if the requirement (1) in the next lemma is strengthened as $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r$.

Lemma 5.20. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r$ then there exists a stable context $D_{\widetilde{X}}$ such that

(1) $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r' \Rrightarrow r$ for some r', and
(2) $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any $\widetilde{q}$ with $|\widetilde{q}| = |\widetilde{X}|$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^n|r$. It proceeds by induction on n. For the inductive base $n = 0$, it follows from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv r \stackrel{\tau}{\not\longrightarrow}$ that $C_{\widetilde{X}}$ is stable by Lemma 5.7. Then it is straightforward to verify that $C_{\widetilde{X}}$ itself is exactly what we seek. For the inductive step, assume $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} t(\stackrel{\tau}{\longrightarrow})^k|r$ for some t. Then, for the τ-labelled transition

$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} t$, either the clause (1) or (2) in Lemma 5.6 holds. For the first alternative, it is easy to handle and omitted. Next we consider the second alternative.

In such situation, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i_0 \leq |\widetilde{X}|$ that satisfy (P-τ-1) – (P-τ-4). By (P-τ-2), we have

$$t \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \text{ for some } p' \text{ with } p_{i_0} \xrightarrow{\tau} p'.$$

Then, for $C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}(\xrightarrow{\tau})^k|r$, by IH, there exists a stable context $D'_{\widetilde{X},Z}$ such that

$$C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \stackrel{\epsilon}{\Longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \stackrel{\epsilon}{\Longrightarrow} |r' \Rrightarrow r \text{ for some } r' \tag{5.20.1}$$

and for any q' and $\widetilde{q}$, we have

$$C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\} \stackrel{\epsilon}{\Longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}. \tag{5.20.2}$$

In particular, we have $C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \stackrel{\epsilon}{\Longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\}$ where distinct visible actions $\widetilde{a_X}$ and a_Z are fresh. For this transition, applying Lemma 5.6 finitely times (notice that, in this procedure, since $\widetilde{a_X.0}$ and $a_Z.0$ are stable, the clause (2) in Lemma 5.6 is always false), then by clause (1) in Lemma 5.6, we get the sequence below

$$\begin{aligned} C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \equiv C^0_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \xrightarrow{\tau} C^1_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \xrightarrow{\tau} \\ \cdots \xrightarrow{\tau} C^n_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \equiv D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\}. \end{aligned}$$

Here $n \geq 0$ and $C^i_{\widetilde{X},Z}$ satisfies (C-τ-1,2,3) for each $1 \leq i \leq n$. Moreover, since Z is 1-active in $C''_{\widetilde{X},Z}$, by (C-τ-3-i), so is Z in $C^n_{\widetilde{X},Z}$. On the other hand, by Lemma 5.19, we have $C^n_{\widetilde{X},Z} \equiv D'_{\widetilde{X},Z}$. Hence we can make the conclusion below

$$Z \text{ is 1-active in } D'_{\widetilde{X},Z}. \tag{5.20.3}$$

Since $C'_{\widetilde{X}}$ and $C''_{\widetilde{X},Z}$ satisfy (P-τ-1) and (P-τ-3), for any $\widetilde{s}$, we get

$$C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \equiv C''_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s_{i_0}/Z\}. \tag{5.20.4}$$

In order to complete the proof, it suffices to find a stable context $D_{\widetilde{X}}$ satisfying conditions (1) and (2). In the following, we shall use $\widetilde{a_X.0}$ again to obtain such context.

Since $\widetilde{a_X.0}$ and $D'_{\widetilde{X},Z}$ are stable, by (5.20.2), we get

$$C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\} \stackrel{\epsilon}{\Longrightarrow} |D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}.$$

Moreover, by (5.20.4), we have $C'_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \equiv C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}$. Thus, it follows that

$$C'_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}.$$

Then, since $\widetilde{a_X.0}$ are stable, by Lemma 5.17, there exists a stable context $B_{\widetilde{X}}$ such that

$$B_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \equiv D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\} \tag{5.20.5}$$

and

$$C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} B_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \text{ for any } \widetilde{s}. \tag{5.20.6}$$

On the other hand, by (5.20.4) and Lemma 5.13, we have $C_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |t' \Rrightarrow D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}$ for some t'. Further, since $\widetilde{a_X.0}$ are stable, by Lemma 5.17, there exists a stable context $D_{\widetilde{X}}$ such that

$$t' \equiv D_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\} \tag{5.20.7}$$

and

$$C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \text{ for any } \widetilde{s}. \tag{5.20.8}$$

Notice that, (5.20.8) comes from (MS-τ-3-ii) with $\widetilde{Y} = \emptyset$. In the following, we intend to prove that $D_{\widetilde{X}}$ is what we seek. It immediately follows from (5.20.8) that $D_{\widetilde{X}}$ meets the requirement (2). We are left with the task of verifying that $D_{\widetilde{X}}$ satisfies the condition (1). So far, for any $\widetilde{s}$, we have the diagram below, where the first line comes from (5.20.4),

$$\begin{array}{ccccc}
C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \Rrightarrow & C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \equiv & C''_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s_{i_0}/Z\} \\
\Downarrow\epsilon \text{ by } (5.20.8) & & \Downarrow\epsilon \text{ by } (5.20.6) & & \Downarrow\epsilon \text{ by } (5.20.2) \\
D_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \Rrightarrow & B_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \equiv & D'_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s_{i_0}/Z\}
\end{array}$$

Here the last line in the above follows from (5.20.7) and (5.20.5) using Lemma 5.19. Further, by Lemma 5.4 and $p_{i_0} \stackrel{\tau}{\longrightarrow} p'$, it follows from (5.20.1) and (5.20.3) that

$$B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv D'_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p_{i_0}/Z\} \stackrel{\tau}{\longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \stackrel{\epsilon}{\Longrightarrow} |r' \Rrightarrow r.$$

Finally, since $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, by Lemma 5.13, we get $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r'' \Rrightarrow r' \Rrightarrow r$ for some r'', which, together with $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, implies that the stable context $D_{\widetilde{X}}$ also meets the requirement (1), as desired. □

The result below asserts that there exist another "canonical"evolution paths from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ to a given stable τ-descendant r. For these paths, an unstable p_i evolves first provided that such p_i is located in an active position.

Lemma 5.21. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$ and X_i is 1-active in $C_{\widetilde{X}}$ for some $i \leq |\widetilde{X}|$, then there exists p' such that $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^n|q$ for some $n \geq 0$. We shall prove it by induction on n. For the inductive base $n = 0$, we have $p_i \stackrel{\tau}{\not\longrightarrow}$ by Lemma 5.4, and hence it holds trivially by taking $p' \equiv p_i$. For the inductive step $n = k + 1$, suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} r(\stackrel{\tau}{\longrightarrow})^k|q$ for some r. For the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} r$, either the clause (1) or (2) in Lemma 5.6 holds.

For the first alternative, there exists a context $C'_{\widetilde{X}}$ such that

(1.1) X_i is 1-active in $C'_{\widetilde{X}}$ (by (C-τ-3-i)),

(1.2) $r \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, and

(1.3) $C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\}$ for any $\widetilde{s}$.

By (1.1), we can apply induction hypothesis for the transition $r \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k|q$, and hence there exists p' such that $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$ and $C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$. On the other hand, since X_i is 1-active in $C_{\widetilde{X}}$ and $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$, we have $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\}$ by Lemma 5.4. Moreover, $C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\}$ by (1.3). Therefore, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$, as desired.

For the second alternative, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ and $i_0 \leq |\widetilde{X}|$ such that

(2.1) Z is 1-active in $C''_{\widetilde{X},Z}$,

(2.2) $r \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'_{i_0}/Z\}$ for some p'_{i_0} with $p_{i_0} \stackrel{\tau}{\longrightarrow} p'_{i_0}$, and

(2.3) $C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s'/Z\}$ for any $\widetilde{s}$ and s' with $s_{i_0} \stackrel{\tau}{\longrightarrow} s'$.

For the case $i_0 = i$, we have $C_{\widetilde{X}} \equiv C'_{\widetilde{X}}$ by (P-τ-1), and hence $r \equiv C_{\widetilde{X}}\{\widetilde{p}[p'_{i_0}/p_i]/\widetilde{X}\}$ by (2.2) and (P-τ-3). For the transition $r \equiv C_{\widetilde{X}}\{\widetilde{p}[p'_{i_0}/p_i]/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k|q$, by IH, there exists p'' such that $p'_{i_0} \stackrel{\epsilon}{\Longrightarrow} |p''$ and $C_{\widetilde{X}}\{\widetilde{p}[p'_{i_0}/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p''/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$. Hence $p_{i_0} \stackrel{\tau}{\longrightarrow} p'_{i_0} \stackrel{\epsilon}{\Longrightarrow} |p''$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p'_{i_0}/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p''/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$.

Next we consider another case $i_0 \neq i$. Then for the transition $r \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'_{i_0}/Z\}(\stackrel{\tau}{\longrightarrow})^k|q$, by IH, there exists p' such that $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$ and

$$C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'_{i_0}/Z\} \stackrel{\epsilon}{\Longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{p}[p'/p_i]/\widetilde{X}, p'_{i_0}/Z\} \stackrel{\epsilon}{\Longrightarrow} |q.$$

On the other hand, since X_i is 1-active in $C_{\widetilde{X}}$ and $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$, by Lemma 5.4, we obtain $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\}$. Moreover, $C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{p}[p'/p_i]/\widetilde{X}, p'_{i_0}/Z\}$ by (2.3). Thus

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{p}[p'/p_i]/\widetilde{X}, p'_{i_0}/Z\} \stackrel{\epsilon}{\Longrightarrow} |q,$$

as desired. □

6. Precongruence

This section intends to establish a fundamental property that $\sqsubseteq_{RS}$ is a precongruence, that is, it is substitutive w.r.t all operations in CLL_R. This constitutes one of two main results of this paper. Its proof is far from trivial and requires a solid effort. As mentioned in Section 1, a distinguishing feature of LLTS is that it involves consideration of inconsistencies. It is the inconsistency predicate F that make everything become quite troublesome. A crucial part in carrying out the proof is that we need to prove that $C_X\{q/X\} \in F$ implies $C_X\{p/X\} \in F$ whenever $p \sqsubseteq_{RS} q$. Its argument will be divided into two steps. First, we shall show that, for any stable process p, $C_X\{\tau.p/X\}$ is in accordance with $C_X\{p/X\}$ in the consistency. Second, we intend to prove that $C_X\{q/X\} \in F$ implies $C_X\{p/X\} \in F$ whenever p and q are uniform w.r.t stability and $p \sqsubseteq_{RS} q$.

Definition 6.1 (Uniform w.r.t stability). Given two tuples $\widetilde{p}$ and $\widetilde{q}$ with $|\widetilde{q}| = |\widetilde{p}|$, they are said to be uniform w.r.t stability, in symbols $\widetilde{p} \bowtie \widetilde{q}$, if they are component-wise uniform w.r.t stability, that is, p_i is stable iff q_i is stable for each $i \leq |\widetilde{p}|$.

An elementary property of this notion is given below.

Lemma 6.1. The uniformity w.r.t stability are preserved under substitutions. That is, for any $\widetilde{p}$, $\widetilde{q}$ and $C_{\widetilde{X}}$, if $\widetilde{p} \bowtie \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \bowtie C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$.

Proof. Immediately follows from Lemma 5.6. □

Notation For convenience, given tuples $\widetilde{p}$ and $\widetilde{q}$, for $R \in \{\sqsubseteq_{RS}, \underset{\sim}{\sqsubset}_{RS}, \overset{\epsilon}{\Longrightarrow}|, \equiv\}$, the notation $\widetilde{p}R\widetilde{q}$ means that $|\widetilde{p}| = |\widetilde{q}|$ and $p_i R q_i$ for each $i \leq |\widetilde{p}|$.

Lemma 6.2. For any $C_{\widetilde{X}}$, $\widetilde{p}$ and $\widetilde{q}$ with $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ are stable and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a}$ iff $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a}$ for any $a \in Act$.

Proof. We give the proof only for the implication from right to left, the same argument applies to the other implication. Assume $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} q'$. Then there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \emptyset$ that satisfy (CP-a-1) – (CP-a-4) in Lemma 5.9. Hence, due to (CP-a-1) and (CP-a-3-i), there exist $i_Y \leq |\widetilde{X}|$ with $Y \in \widetilde{Y}$ such that for any $\widetilde{r}$ with $|\widetilde{r}| = |\widetilde{X}|$

$$C_{\widetilde{X}}\{\widetilde{r}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{r}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{r}/\widetilde{X}, \widetilde{r_{i_Y}}/\widetilde{Y}\}. \tag{6.2.1}$$

In particular, by Lemma 5.12, it follows from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}$ that both $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}$ and $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q_{i_Y}}/\widetilde{Y}\}$ are stable. Then, for each $Y \in \widetilde{Y}$, both p_{i_Y} and q_{i_Y} are stable by Lemma 5.4 and (CP-a-2). Moreover, by (6.2.1) with $\widetilde{r} \equiv \widetilde{p}$ and Lemma 5.15 and 5.5, we have $p_{i_Y} \notin F$ with $Y \in \widetilde{Y}$ due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$. Therefore, for each $Y \in \widetilde{Y}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that $p_{i_Y} \underset{\sim}{\sqsubset}_{RS} q_{i_Y}$, and $\mathcal{I}(p_{i_Y}) = \mathcal{I}(q_{i_Y})$ because of $p_{i_Y} \notin F$. Hence $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a}$ by (CP-a-3-iii). □

In the following, we intend to show that, for any stable p, $C_X\{p/X\}$ and $C_X\{\tau.p/X\}$ are undifferentiated w.r.t consistency, which falls naturally into two parts: Lemma 6.3 and 6.5.

Lemma 6.3. For any C_X and stable p, $C_X\{p/X\} \notin F$ implies $C_X\{\tau.p/X\} \notin F$.

Proof. Let p be any stable process. Set

$$\Omega \triangleq \{B_X\{\tau.p/X\} : B_X\{p/X\} \notin F \text{ and } B_X \text{ is a context}\}.$$

Similar to Lemma 4.4, it suffices to prove that for any $t \in \Omega$, each proof tree of tF has a proper subtree with the root labelled with sF for some $s \in \Omega$. Suppose that $C_X\{\tau.p/X\} \in \Omega$ and $\mathcal{T}$ is any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{\tau.p/X\}F$. Hence $C_X\{p/X\} \notin F$. We distinguish six cases based on the form of C_X.

Case 1 C_X is closed or $C_X \equiv X$.

In such situation, it is easy to see that $C_X\{\tau.p/X\} \notin F$. Hence there is no proof tree of $C_X\{\tau.p/X\}F$. Thus the conclusion holds trivially.

Case 2 $C_X \equiv \alpha.B_X$.

Then the last rule applied in $\mathcal{T}$ is $\frac{B_X\{\tau.p/X\}F}{\alpha.B_X\{\tau.p/X\}F}$. Since $C_X\{p/X\} \notin F$, we get $B_X\{p/X\} \notin F$. Hence $B_X\{\tau.p/X\} \in \Omega$, moreover, the node directly above the root of $\mathcal{T}$ is labelled with $B_X\{\tau.p/X\}F$, as desired.

Case 3 $C_X \equiv B_X \vee D_X$.

Clearly, the last rule applied in $\mathcal{T}$ is $\frac{B_X\{\tau.p/X\}F,D_X\{\tau.p/X\}F}{B_X\{\tau.p/X\}\vee D_X\{\tau.p/X\}F}$. Since $C_X\{p/X\} \notin F$, either $B_X\{p/X\} \notin F$ or $D_X\{p/X\} \notin F$. W.l.o.g, assume $B_X\{p/X\} \notin F$. Then $B_X\{\tau.p/X\} \in \Omega$. Moreover, it is obvious that $\mathcal{T}$ has a proper subtree with the root labelled with $B_X\{\tau.p/X\}F$.

Case 4 $C_X \equiv B_X \odot D_X$ with $\odot \in \{\Box, \|_A\}$.

W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{B_X\{\tau.p/X\}F}{B_X\{\tau.p/X\}\odot D_X\{\tau.p/X\}F}$. It is evident that $B_X\{p/X\} \notin F$ due to $C_X\{p/X\} \notin F$. Hence $B_X\{\tau.p/X\} \in \Omega$, as desired.

Case 5 $C_X \equiv \langle Y|E\rangle$.

Then the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{\tau.p/X\}F}{\langle Y|E\rangle\{\tau.p/X\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{\tau.p/X\}F}.$$

For the first alternative, since $C_X\{p/X\} \equiv \langle Y|E\rangle\{p/X\} \notin F$, by Lemma 4.1(8), we get $\langle t_Y|E\rangle\{p/X\} \notin F$. Hence $\langle t_Y|E\rangle\{\tau.p/X\} \in \Omega$.

For the second alternative, since $C_X\{p/X\} \notin F$, we get $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q$ for some q. Moreover, by Lemma 5.17, it follows from $p \stackrel{\tau}{\not\longrightarrow}$ that there exists a stable context C'_X such that

$$q \equiv C'_X\{p/X\} \text{ and } C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_X\{\tau.p/X\}. \tag{6.3.1}$$

Further, by Lemma 5.18 and $\tau.p \stackrel{\tau}{\longrightarrow} |p$, we get

$$C'_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |s \text{ for some } s. \tag{6.3.2}$$

For the above transition, by Lemma 5.17 again, there exists $C''_{X,\widetilde{Z}}$ with $X \notin \widetilde{Z}$ such that

$$s \equiv C''_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \text{ and } C'_X\{p/X\} \Rrightarrow C''_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\}.$$

Thus, by Lemma 5.15, we have $C''_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\} \notin F$ because of $q \equiv C'_X\{p/X\} \notin F$. Set

$$C'''_X \triangleq C''_{X,\widetilde{Z}}\{p/\widetilde{Z}\}.$$

Then it follows from $C'''_X\{p/X\} \equiv C''_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\} \notin F$ that $s \equiv C'''_X\{\tau.p/X\} \in \Omega$. Moreover, $\mathcal{T}$ contains a proper subtree with the root labelled with sF due to (6.3.1) and (6.3.2).

Case 6 $C_X \equiv B_X \wedge D_X$.

Clearly, the last rule applied in $\mathcal{T}$ has one of the following formats.

Case 6.1 $\frac{B_X\{\tau.p/X\}F}{B_X\{\tau.p/X\}\wedge D_X\{\tau.p/X\}F}$.

Similar to Case 4, omitted.

Case 6.2 $\frac{B_X\{\tau.p/X\}\xrightarrow{a}r,\, D_X\{\tau.p/X\}\not\xrightarrow{a},\, B_X\{\tau.p/X\}\wedge D_X\{\tau.p/X\}\not\xrightarrow{\tau}}{B_X\{\tau.p/X\}\wedge D_X\{\tau.p/X\}F}$.

In such situation, $B_X\{\tau.p/X\}$, C_X and B_X are stable. Moreover, since p is stable, so is $B_X\{p/X\}$. Due to $C_X\{p/X\} \notin F$, we obtain $B_X\{p/X\} \notin F$. Then, by Lemma 6.2, it follows from $p =_{RS} \tau.p$ and $B_X\{\tau.p/X\} \xrightarrow{a}$ that

$$B_X\{p/X\} \xrightarrow{a} . \tag{6.3.3}$$

Similarly, it comes from $D_X\{\tau.p/X\} \not\xrightarrow{a}$ that

$$D_X\{p/X\} \not\xrightarrow{a} . \tag{6.3.4}$$

On the other hand, since $B_X \wedge D_X$ and p are stable, so is $B_X\{p/X\} \wedge D_X\{p/X\}$. So, by (6.3.3) and (6.3.4), we get $C_X\{p/X\} \equiv B_X\{p/X\} \wedge D_X\{p/X\} \in F$ by the rule Rp_{10}, which contradicts that $C_X\{\tau.p/X\} \in \Omega$. Hence this case is impossible.

Case 6.3 $\frac{C_X\{\tau.p/X\}\xrightarrow{\alpha}s,\, \{rF : C_X\{\tau.p/X\}\xrightarrow{\alpha}r\}}{C_X\{\tau.p/X\}F}$.

The argument splits into two cases based on α.

Case 6.3.1 $\alpha = \tau$.

We distinguish two cases depending on whether C_X is stable.

Case 6.3.1.1 C_X is not stable.

Since $C_X\{p/X\} \notin F$, we have $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p'. Moreover, by Lemma 5.20, there exist p'' and stable C^*_X such that

$$C_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{p/X\} \overset{\epsilon}{\Longrightarrow} |p'' \Rrightarrow p'$$

and

$$C_X\{t/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{t/X\} \text{ for any } t.$$

Further, since C_X is not stable and $p \not\xrightarrow{\tau}$, by Lemma 5.6, there exists C'_X such that

$$C_X\{p/X\} \xrightarrow{\tau} C'_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{p/X\} \text{ and } C_X\{\tau.p/X\} \xrightarrow{\tau} C'_X\{\tau.p/X\}.$$

Since $p' \notin F$ and $p'' \Rrightarrow p'$, by Lemma 5.15, we get $p'' \notin F$. Together with the transitions $C'_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{p/X\} \overset{\epsilon}{\Longrightarrow} |p''$, by Lemma 4.2, this implies $C'_X\{p/X\} \notin F$. Hence $C'_X\{\tau.p/X\} \in \Omega$, and $\mathcal{T}$ has a proper subtree with the root labelled with $C'_X\{\tau.p/X\}F$.

Case 6.3.1.2 C_X is stable.

Due to $C_X\{\tau.p/X\} \xrightarrow{\tau} s$, either the clause (1) or (2) in Lemma 5.6 holds. Since C_X is stable, by (C-τ-2) in Lemma 5.6, it is easy to see that the clause (1) does not hold, and hence the clause (2) holds, that is, there exists $C'_{X,Z}$ with $X \neq Z$ such that

$$C_X\{\tau.p/X\} \xrightarrow{\tau} C'_{X,Z}\{\tau.p/X, p/Z\} \text{ and } C_X\{p/X\} \Rrightarrow C'_{X,Z}\{p/X, p/Z\}.$$

Set

$$C''_X \triangleq C'_{X,Z}\{p/Z\}.$$

Hence $\mathcal{T}$ has a proper subtree with the root labelled with $C''_X\{\tau.p/X\}F$. On the other hand, by Lemma 5.15, it follows from $C_X\{p/X\} \notin F$ that $C'_{X,Z}\{p/X, p/Z\} \notin F$. Thus $C''_X\{\tau.p/X\} \equiv C'_{X,Z}\{\tau.p/X, p/Z\} \in \Omega$, as desired.

Case 6.3.2 $\alpha \in Act$.

Then it is not difficult to know that both C_X and $C_X\{p/X\}$ are stable. Moreover, since $C_X\{\tau.p/X\} \xrightarrow{\alpha}$, $\tau.p =_{RS} p$ and $C_X\{p/X\} \notin F$, by Lemma 6.2, we get $C_X\{p/X\} \xrightarrow{\alpha}$. Further, by Theorem 4.2, it follows from $C_X\{p/X\} \notin F$ that $C_X\{p/X\} \xrightarrow{\alpha}_F q$ for some q. For such α-labelled transition, by Lemma 5.9, there exist C'_X, $C'_{X,\widetilde{Z}}$ and $C''_{X,\widetilde{Z}}$ with $X \notin \widetilde{Z}$ that realize (CP-a-1) – (CP-a-4).

In order to complete the proof, we intend to prove that $\widetilde{Z} = \emptyset$. On the contrary, suppose $\widetilde{Z} \neq \emptyset$. Then, by (CP-$a$-2) and (CP-$a$-3-i), there exists an active occurrence of the variable X in C'_X. So, by Lemma 5.4, $C'_X\{\tau.p/X\} \xrightarrow{\tau}$. Then, by Lemma 5.12, it follows from $C_X\{\tau.p/X\} \Rrightarrow C'_X\{\tau.p/X\}$ (i.e., (CP-a-1)) that $C_X\{\tau.p/X\} \xrightarrow{\tau}$, which contradicts $C_X\{\tau.p/X\} \xrightarrow{\alpha}$.

Thus $\widetilde{Z} = \emptyset$, and hence $q \equiv C''_{X,\widetilde{Z}}\{p/X\}$ by (CP-a-3-ii). Since $C_X\{\tau.p/X\}$ is stable, by (CP-a-3-iii), we get $C_X\{\tau.p/X\} \xrightarrow{\alpha} C''_{X,\widetilde{Z}}\{\tau.p/X\}$. Thus, $\mathcal{T}$ contains a proper subtree with the root labelled with $C''_{X,\widetilde{Z}}\{\tau.p/X\}F$, moreover, $C''_{X,\widetilde{Z}}\{\tau.p/X\} \in \Omega$ due to $C''_{X,\widetilde{Z}}\{p/X\} \equiv q \notin F$.

Case 6.4 $\frac{\{rF : B_X\{\tau.p/X\} \wedge D_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{B_X\{\tau.p/X\} \wedge D_X\{\tau.p/X\}F}$.

Analogous to the second alternative in Case 5, omitted. □

In order to show the converse of the above result, the preliminary result below is given. Here, for any finite set S of processes, by virtue of the commutative and associative laws of external choice (Zhang *et al.* 2011), we may introduce the notation of a generalized external choice (denoted by $\mathop{\Box}\limits_{p \in S} p$) by the standard method.

Lemma 6.4. Let t_1, t_2 be two terms and $\{X\} \cup \widetilde{Z}$ a tuple of variables such that none of recursive variable occurring in t_i(with $i = 1, 2$) is in $\{X\} \cup \widetilde{Z}$. Suppose that Z is active in t_1, t_2 for each $Z \in \widetilde{Z}$ and

$$T \triangleq \begin{cases} \mathop{\Box}\limits_{Z \in \widetilde{Z}} \alpha.a_Z.0 & \text{if } \widetilde{Z} \neq \emptyset \\ a_X.0 & \text{otherwise} \end{cases}$$

where a_X and $\widetilde{a_Z}$ are distinct fresh visible actions and $\alpha \in Act$. Then

(1) if $t_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ then $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$ for any p and $\widetilde{q}$;

(2) if $t_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Rrightarrow_1 t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ then $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \Rrightarrow_1 t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$ for any p and $\widetilde{q}$;

Proof. **(1)** It proceeds by induction on t_1. We distinguish three cases as follows.

Case 1 t_1 is closed or t_1 is of the format X or $\beta.s$ or $s_1 \vee s_2$ or $\langle Y|E\rangle$.

Since Z is active in t_1 for each $Z \in \widetilde{Z}$, we get $\widetilde{Z} = \emptyset$. Then it follows by Lemma 5.19.

Case 2 $t_1 \equiv s_1 \odot s_2$ with $\odot \in \{\|_A, \wedge\}$.

Then $t_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \odot s_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$. Since neither $\widetilde{a_Z.0}$ nor T contain $\odot$, there exist s_1', s_2' such that $s_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_1'\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$, $s_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_2'\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ and $t_2 \equiv s_1' \odot s_2'$. Hence it immediately follows that $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$ for any p and $\widetilde{q}$ by IH.

Case 3 $t_1 \equiv s_1 \Box s_2$.

Then $t_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Box s_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$. Hence the most top operator of $t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ is an external choice $\Box$. Clearly, such operator comes from either T or t_2. For the former, we get $t_2 \equiv X$ and hence $\widetilde{Z} = \emptyset$. Thus $t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}(\equiv a_X.0)$ does not contain the operator $\Box$ at all, a contradiction. So this case is impossible. Hence $t_2 \equiv s_1' \Box s_2'$ for some s_1' and s_2', and the rest of the proof runs as Case 2.

(2) If $FV(t_1) \cap \widetilde{Z} = \emptyset$, it follows by Lemma 5.19. Next we consider another case $FV(t_1) \cap \widetilde{Z} \neq \emptyset$. It proceeds by induction on t_1. Since Z is active in t_1 for each $Z \in \widetilde{Z}$, we get either $t_1 \equiv Z$ or $t_1 \equiv s_1 \odot s_2$ for some s_1 and s_2, where $Z \in \widetilde{Z}$ and $\odot \in \{\wedge, \|_A, \Box\}$. We give the proof only for the case $t_1 \equiv s_1 \Box s_2$, the proofs for the remaining cases are straightforward and omitted.

It follows from $t_1 \equiv s_1 \Box s_2$ that

$$t_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Box s_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Rrightarrow_1 t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}.$$

So the most top operator of $t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ is an external choice $\Box$ which comes from either T or t_2. Similar to Case 3 in the proof for item (1), we can make the conclusion that there exist s_1', s_2' such that $t_2 \equiv s_1' \Box s_2'$. Moreover, it is easily seen that either $s_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ or $s_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ triggers the unfolding from $t_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ to $t_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$. W.l.o.g, we consider the first alternative. Then $s_1\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Rrightarrow_1 s_1'\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ and $s_2\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_2'\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$. Hence, by IH and item (1), for any p and $\widetilde{q}$, we have $s_1\{p/X, \widetilde{q}/\widetilde{Z}\} \Rrightarrow_1 s_1'\{p/X, \widetilde{q}/\widetilde{Z}\}$ and $s_2\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv s_2'\{p/X, \widetilde{q}/\widetilde{Z}\}$. Therefore, $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv s_1\{p/X, \widetilde{q}/\widetilde{Z}\} \Box s_2\{p/X, \widetilde{q}/\widetilde{Z}\} \Rrightarrow_1 t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$. □

The next lemma establishes the converse of Lemma 6.3.

Lemma 6.5. For any C_X and stable process p, $C_X\{\tau.p/X\} \notin F$ implies $C_X\{p/X\} \notin F$.

Proof. Let p be any stable process. Set

$$\Omega \triangleq \{B_X\{p/X\} : B_X\{\tau.p/X\} \notin F \text{ and } B_X \text{ is a context}\}.$$

Assume $t \in \Omega$. Then $t \equiv C_X\{p/X\}$ for some C_X such that $C_X\{\tau.p/X\} \notin F$. Let $\mathcal{T}$ be any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{p/X\}F$. Similar to Lemma 6.3, it is enough to prove that $\mathcal{T}$ has a proper subtree with the root labelled with sF for some

$s \in \Omega$, which is a routine case analysis based on the last rule applied in $\mathcal{T}$. Here we treat only three primary cases.

Case 1 $\frac{\{rF : C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{C_X\{p/X\}F}$ with $C_X \equiv \langle Y|E\rangle$.

Since $C_X\{\tau.p/X\} \notin F$, we get $C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q$ for some q. By Lemma 5.17, for this transition, there exists a stable context $C'_{X,\widetilde{Z}}$ satisfying (MS-τ-1) – (MS-τ-7). In particular, since p and q are stable, by (MS-τ-2,7), we have

$$q \equiv C'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \notin F.$$

Moreover, since each $Z(\in \widetilde{Z})$ is 1-active in $C'_{X,\widetilde{Z}}$ (i.e., (MS-τ-1)) and $\tau.p \stackrel{\tau}{\longrightarrow} p$, by Lemma 5.4, we get $C'_{X,\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \equiv q \notin F$, which, by Lemma 4.2, implies

$$C'_{X,\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Z}\} \notin F. \tag{6.5.1}$$

Let a_X be any fresh visible action. By (MS-τ-3-i), it follows from $a_X.0 \stackrel{\epsilon}{\Longrightarrow} |a_X.0$ that there exists s such that

$$C_X\{a_X.0/X\} \stackrel{\epsilon}{\Longrightarrow} s \Rrightarrow C'_{X,\widetilde{Z}}\{a_X.0/X, a_X.0/\widetilde{Z}\}. \tag{6.5.2}$$

Since $a_X.0$ and $C'_{X,\widetilde{Z}}$ are stable, so is $C'_{X,\widetilde{Z}}\{a_X.0/X, a_X.0/\widetilde{Z}\}$ by Lemma 5.6. Then, by Lemma 5.12, s is stable. Thus, for the transition in (6.5.2), by Lemma 5.17, there exists a stable context C^*_X such that

$$s \equiv C^*_X\{a_X.0/X\} \text{ and } C_X\{r/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{r/X\} \text{ for any } r. \tag{6.5.3}$$

Then, by Lemma 5.19, it follows from $s \equiv C^*_X\{a_X.0/X\} \Rrightarrow C'_{X,\widetilde{Z}}\{a_X.0/X, a_X.0/\widetilde{Z}\}$ that

$$C^*_X\{\tau.p/X\} \Rrightarrow C'_{X,\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Z}\}.$$

Hence $C^*_X\{\tau.p/X\} \notin F$ by (6.5.1) and Lemma 5.15, which implies $C^*_X\{p/X\} \in \Omega$. On the other hand, since C^*_X and p are stable, so is $C^*_X\{p/X\}$ by Lemma 5.6. Moreover, by (6.5.3), we get $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |C^*_X\{p/X\}$. Therefore, $\mathcal{T}$ has a proper subtree with the root labelled with $C^*_X\{p/X\}F$.

Case 2 $\frac{B_X\{p/X\} \stackrel{a}{\longrightarrow} r,\, D_X\{p/X\} \stackrel{a}{\not\longrightarrow},\, C_X\{p/X\} \stackrel{\tau}{\not\longrightarrow}}{C_X\{p/X\}F}$ with $C_X \equiv B_X \wedge D_X$.

Clearly, in such situation, both B_X and D_X are stable. Since $C_X\{\tau.p/X\} \notin F$, we have $C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q$ for some q. So, there exist s and t such that $q \equiv s \wedge t$ and

$$B_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ and } D_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |t.$$

Then, for these two transitions, by Lemma 5.17, there exist $B'_{X,\widetilde{Y}}$ and $D'_{X,\widetilde{Z}}$ satisfying (MS-τ-1) – (MS-τ-7) respectively. In particular, since p, B_X and D_X are stable, by (MS-τ-2,4,7), we have

(1) $s \equiv B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ and $B_X\{p/X\} \Rrightarrow B'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}$;

(2) $t \equiv D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\}$ and $D_X\{p/X\} \Rrightarrow D'_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\}$.

Hence, by Lemma 5.12, it follows from $B_X\{p/X\} \xrightarrow{a}$ and $D_X\{p/X\} \not\xrightarrow{a}$ that

$$B'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \xrightarrow{a} \text{ and } D'_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\} \not\xrightarrow{a} .$$

Further, since $B'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}$ and $B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ are stable, by Lemma 6.2, it follows from $\tau.p =_{RS} p$ and $s \equiv B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \notin F$ that $B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \xrightarrow{a}$. Similarly, we also have $D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \not\xrightarrow{a}$. Hence $q \equiv s \wedge t \in F$ by the rule Rp_{10}, a contradiction. Thus this case is impossible.

Case 3 $\frac{C_X\{p/X\} \xrightarrow{\alpha} r', \{rF: C_X\{p/X\} \xrightarrow{\alpha} r\}}{C_X\{p/X\}F}$ with $C_X \equiv B_X \wedge D_X$.
Since $C_X\{\tau.p/X\} \notin F$, we have

$$C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q \text{ for some } q. \tag{6.5.4}$$

Next we distinguish two cases based on α.

Case 3.1 $\alpha = \tau$.
By (6.5.4) and Lemma 5.20, there exist t and stable context C^*_X such that

$$C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |t \Rrightarrow q \notin F$$

and

$$C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{p/X\}.$$

Moreover, since $p \not\xrightarrow{\tau}$ and $\tau \in \mathcal{I}(C_X\{p/X\})$, by Lemma 5.6, there exists a context C'_X such that

$$C_X\{p/X\} \xrightarrow{\tau} C'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{p/X\}$$

and

$$C_X\{\tau.p/X\} \xrightarrow{\tau} C'_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |t.$$

Further, by Lemma 5.15, it follows from $q \notin F$ and $t \Rrightarrow q$ that $t \notin F$. Then, by Lemma 4.2 and the transition above, we have $C'_X\{\tau.p/X\} \notin F$. Hence $C'_X\{p/X\} \in \Omega$ and one of nodes directly above the root of $\mathcal{T}$ is labelled with $C'_X\{p/X\}F$, as desired.

Case 3.2 $\alpha \in Act$.
In this case, C_X is stable by Lemma 5.7. By (6.5.4) and Lemma 5.17, there exists a stable context $C'_{X,\widetilde{Y}}$ with $X \notin \widetilde{Y}$ that satisfies (MS-τ-1) – (MS-τ-7). Then $q \equiv C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ due to $p \not\xrightarrow{\tau}$ and (MS-τ-2). Moreover, since C_X is stable, by (MS-τ-4), we have

$$C_X\{r/X\} \Rrightarrow C'_{X,\widetilde{Y}}\{r/X, r/\widetilde{Y}\} \text{ for any } r. \tag{6.5.5}$$

Then, by $C_X\{p/X\} \xrightarrow{\alpha}$ and Lemma 5.12, we get

$$C'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \xrightarrow{\alpha} . \tag{6.5.6}$$

Further, by Lemma 6.2, we also have $C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \xrightarrow{\alpha}$ because of $\tau.p =_{RS} p$ and

$q \equiv C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \notin F$. Thus, by Theorem 4.2, we obtain

$$C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \xrightarrow{\alpha}_F t \text{ for some } t.$$

For such α-labelled transition, by Lemma 5.9, there exist $C''_{X,\widetilde{Y}}$, $C''_{X,\widetilde{Y},\widetilde{Z}}$ and $C'''_{X,\widetilde{Y},\widetilde{Z}}$ with $(\{X\} \cup \widetilde{Y}) \cap \widetilde{Z} = \emptyset$ that realize (CP-a-1) – (CP-a-4). In particular, due to $\tau.p \not\xrightarrow{\alpha}$ and (CP-a-3-ii), there exist p'_Z with $Z \in \widetilde{Z}$ such that

$$p \xrightarrow{\alpha} p'_Z \text{ and } t \equiv C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \notin F. \tag{6.5.7}$$

Moreover, by (CP-a-3-iii), for any r, s and s'_Z with $Z \in \widetilde{Z}$ such that $s \xrightarrow{\alpha} s'_Z$, we have

$$C'_{X,\widetilde{Y}}\{r/X, s/\widetilde{Y}\} \xrightarrow{\alpha} C'''_{X,\widetilde{Y},\widetilde{Z}}\{r/X, s/\widetilde{Y}, \widetilde{s'_Z}/\widetilde{Z}\} \text{ whenever } C'_{X,\widetilde{Y}}\{r/X, s/\widetilde{Y}\} \text{ is stable.} \tag{6.5.8}$$

For each $Z \in \widetilde{Z} \cup \{X\}$, we fix a fresh and distinct visible action a_Z and set

$$T \triangleq \begin{cases} \mathop{\Box}\limits_{Z \in \widetilde{Z}} \alpha.a_Z.0, & \text{if } \widetilde{Z} \neq \emptyset; \\ \\ a_X.0, & \text{otherwise.} \end{cases}$$

Since T and $C'_{X,\widetilde{Y}}$ are stable, so is $C'_{X,\widetilde{Y}}\{T/X, T/\widetilde{Y}\}$ by Lemma 5.6. Then, by (6.5.8), we have

$$C'_{X,\widetilde{Y}}\{T/X, T/\widetilde{Y}\} \xrightarrow{\alpha} C'''_{X,\widetilde{Y},\widetilde{Z}}\{T/X, T/\widetilde{Y}, \widetilde{a_Z.0}/\widetilde{Z}\}.$$

So, by Lemma 5.12, it follows from (6.5.5) that there exists t' such that

$$C_X\{T/X\} \xrightarrow{\alpha} t' \text{ and } t' \Rrightarrow C'''_{X,\widetilde{Y},\widetilde{Z}}\{T/X, T/\widetilde{Y}, \widetilde{a_Z.0}/\widetilde{Z}\}. \tag{6.5.9}$$

Then, by Lemma 5.9, it is not difficult to see that there exists a context $B_{X,\widetilde{Z}}$ satisfies the conditions below:

(a) $t' \equiv B_{X,\widetilde{Z}}\{T/X, \widetilde{a_Z.0}/\widetilde{Z}\}$;

(b) none of a_Z with $Z \in \widetilde{Z}$ occurs in $B_{X,\widetilde{Z}}$;

(c) for any s and s'_Z with $Z \in \widetilde{Z}$ such that $s \xrightarrow{\alpha} s'_Z$,

$$C_X\{s/X\} \xrightarrow{\alpha} B_{X,\widetilde{Z}}\{s/X, \widetilde{s'_Z}/\widetilde{Z}\} \text{ whenever } C_X\{s/X\} \text{ is stable.}$$

Now we obtain the diagram below

$$\begin{array}{ccc} & \text{by } (6.5.5) & \\ C_X\{p/X\} & \Rrightarrow & C'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \\ & & \\ \downarrow \alpha \text{ by (c)} & & \downarrow \alpha \text{ by (6.5.6) and (6.5.8)} \\ & \text{by (6.5.9)},\ (a) \text{ and Lemma 6.4} & \\ B_{X,\widetilde{Z}}\{p/X, \widetilde{p'_Z}/\widetilde{Z}\} & \Rrightarrow & C'''_{X,\widetilde{Y},\widetilde{Z}}\{p/X, p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \end{array}$$

By Lemma 6.4, we also have

$$B_{X,\widetilde{Z}}\{\tau.p/X, \widetilde{p'_Z}/\widetilde{Z}\} \Rrightarrow C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\}. \tag{6.5.10}$$

For each $Y \in \widetilde{Y}$, since Y is 1-active in $C'_{X,\widetilde{Y}}$, by Lemma 5.2(1)(2) and $C'_{X,\widetilde{Y}} \Rrightarrow C''_{X,\widetilde{Y}}$ (i.e., (CP-a-1)), so is it in $C''_{X,\widetilde{Y}}$. Moreover, by (CP-a-4-i,ii), for each $Y \in \widetilde{Y} \cap FV(C'''_{X,\widetilde{Y},\widetilde{Z}})$, Y is 1-active in $C'''_{X,\widetilde{Y},\widetilde{Z}}$. Then, by Lemma 5.4, we have

$$C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\}$$

which, together with (6.5.7), implies $C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \notin F$ by Lemma 4.2. Hence, by Lemma 5.15, it follows from (6.5.10) that $B_{X,\widetilde{Z}}\{\tau.p/X, \widetilde{p'_Z}/\widetilde{Z}\} \notin F$. Thus, $B_{X,\widetilde{Z}}\{p/X, \widetilde{p'_Z}/\widetilde{Z}\} \in \Omega$, moreover, $\mathcal{T}$ has a proper subtree with the root labelled with $B_{X,\widetilde{Z}}\{p/X, \widetilde{p'_Z}/\widetilde{Z}\}F$ due to (c) and (6.5.7). $\square$

Hitherto we have completed the first step mentioned at the beginning of this section. Now we return to carry the second step. Before proving Lemma 6.7, a result concerning proof tree is given first.

Lemma 6.6. Let $C_{\widetilde{X},\widetilde{Z}}$ be any context such that for each $Z \in \widetilde{Z}$, Z is active and occurs at most once. If $\widetilde{p}$, $\widetilde{q}$, $\widetilde{t}$, $\widetilde{s}$ and $\widetilde{r}$ are any processes such that

(a) $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$,
(b) $\widetilde{p} \bowtie \widetilde{q}$,
(c) $\widetilde{r} \stackrel{\epsilon}{\Longrightarrow} |\widetilde{t}$,
(d) $\widetilde{s} \underset{\sim RS}{\sqsubset} \widetilde{t}$, and
(e) $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X}, \widetilde{s}/\widetilde{Z}\} \notin F$,

then, for any proof tree $\mathcal{T}$ for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{r}/\widetilde{Z}\}F$, there exist $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, p''_Y and q''_Y with $Y \in \widetilde{Y}$ such that

(1) $\mathcal{T}$ has a subtree with the root labelled with $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{t}/\widetilde{Z}, \widetilde{q''_Y}/\widetilde{Y}\}F$,
(2) $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{s}/\widetilde{Z}, \widetilde{p''_Y}/\widetilde{Y}\} \notin F$, and
(3) $\widetilde{p''_Y} \underset{\sim RS}{\sqsubset} \widetilde{q''_Y}$.

Proof. It proceeds by induction on the depth of $\mathcal{T}$. We distinguish different cases depending on the form of $C_{\widetilde{X},\widetilde{Z}}$.

Case 1 $C_{\widetilde{X},\widetilde{Z}}$ is closed or $C_{\widetilde{X},\widetilde{Z}} \equiv X_i$ or $C_{\widetilde{X},\widetilde{Z}} \equiv Z_j$ for some $i \le |\widetilde{X}|$ and $j \le |\widetilde{Z}|$.

It is straightforward to show that this lemma holds trivially for such case. As a sample, we consider the case $C_{\widetilde{X},\widetilde{Z}} \equiv Z_j$. Since $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X}, \widetilde{s}/\widetilde{Z}\} \equiv s_j \notin F$ and $\widetilde{s} \underset{\sim RS}{\sqsubset} \widetilde{t}$, we have $t_j \notin F$. Hence $r_j \stackrel{\epsilon}{\Longrightarrow}_F |t_j$ by Lemma 4.2. So $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{r}/\widetilde{Z}\} \equiv r_j \notin F$. That is, there is no proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{r}/\widetilde{Z}\}F$. Thus the conclusion holds trivially.

Case 2 $C_{\widetilde{X},\widetilde{Z}}$ is of the format $\alpha.B_{\widetilde{X},\widetilde{Z}}$ or $B_{\widetilde{X},\widetilde{Z}} \vee D_{\widetilde{X},\widetilde{Z}}$ or $\langle Y|E\rangle$.

For these three formats, since each $Z(\in \widetilde{Z})$ is active in $C_{\widetilde{X},\widetilde{Z}}$, it is obvious that

$\widetilde{Z} = \emptyset$. Thus $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \equiv C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}$. So, $\mathcal{T}$ has the root labelled with $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}F$. Therefore, the conclusion holds by setting $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}} \triangleq C_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{Y} = \emptyset$.

Case 3 $C_{\widetilde{X},\widetilde{Z}} \equiv B_{\widetilde{X},\widetilde{Z}} \odot D_{\widetilde{X},\widetilde{Z}}$ with $\odot \in \{\Box, \|_A\}$.
W.l.o.g, assume the last rule applied in $\mathcal{T}$ is

$$\frac{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \odot D_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}.$$

Then $\mathcal{T}$ has a proper subtree $\mathcal{T}'$ with the root labelled with $B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F$. Since $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \notin F$, we get $B_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \notin F$. Then the conclusion immediately follows by applying IH on $\mathcal{T}'$.

Case 4 $C_{\widetilde{X},\widetilde{Z}} \equiv B_{\widetilde{X},\widetilde{Z}} \wedge D_{\widetilde{X},\widetilde{Z}}$.
The argument splits into four cases based on the last rule applied in $\mathcal{T}$.

Case 4.1 $\frac{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\wedge D_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$.
Similar to Case 3, omitted.

Case 4.2 $\frac{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\xrightarrow{a}r',\ D_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\not\xrightarrow{a},\ C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\not\xrightarrow{\tau}}{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\wedge D_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$.
For any $Z(\in \widetilde{Z})$ occurring in $C_{\widetilde{X},\widetilde{Z}}$, since Z is active and $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \not\xrightarrow{\tau}$, by Lemma 5.4, we have $r_Z \not\xrightarrow{\tau}$, and hence $r_Z \equiv t_Z$ because of (c). So, $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \equiv C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}$. Hence $\mathcal{T}$ has the root labelled with $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}F$. Clearly, the conclusion holds by setting $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}} \triangleq C_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{Y} = \emptyset$.

Case 4.3 $\frac{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\xrightarrow{\alpha}s',\ \{rF : C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\xrightarrow{\alpha}r\}}{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$.
If $\alpha \in Act$, the argument is similar to one of Case 4.2 and omitted. In the following, we handle the case $\alpha = \tau$. If $r_Z \not\xrightarrow{\tau}$ for any $Z(\in \widetilde{Z})$ occurring in $C_{\widetilde{X},\widetilde{Z}}$, then the conclusion holds trivially by putting $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}} \triangleq C_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{Y} = \emptyset$. Next we consider another case where $r_{Z_0} \xrightarrow{\tau}$ for some $Z_0(\in \widetilde{Z})$ occurring in $C_{\widetilde{X},\widetilde{Z}}$. Then $r_{Z_0} \xrightarrow{\tau} r' \overset{\epsilon}{\Longrightarrow} |t_{Z_0}$ for some r' by (c), moreover, Z_0 is 1-active in $C_{\widetilde{X},\widetilde{Z}}$. Thus, by Lemma 5.4, we get

$$C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \xrightarrow{\tau} C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}\,[r'/r_{Z_0}]/\widetilde{Z}\}.$$

So, $\mathcal{T}$ has a proper subtree $\mathcal{T}'$ with the root labelled with $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}\,[r'/r_{Z_0}]/\widetilde{Z}\}F$. Since $\widetilde{r}\,[r'/r_{Z_0}] \overset{\epsilon}{\Longrightarrow} |\widetilde{t}$ and $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \notin F$, by IH, $\mathcal{T}'$ has a subtree with the root labelled with $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q''_Y}/\widetilde{Y}\}F$ for some $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, $\widetilde{p''_Y}$ and $\widetilde{q''_Y}$ such that $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p''_Y}/\widetilde{Y}\} \notin F$ and $\widetilde{p''_Y} \sqsubseteq_{RS} \widetilde{q''_Y}$.

Case 4.4 $\frac{\{rF:C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\overset{\epsilon}{\Longrightarrow}|r\}}{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$.

It follows from $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\}\notin F$ that

$$C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\}\overset{\epsilon}{\Longrightarrow}_F|p' \text{ for some } p'.$$

Then, by Lemma 5.17, for such transition, there exist a stable context $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, i_Y and p'''_Y with $Y\in\widetilde{Y}$ that realize (MS-τ-1) – (MS-τ-7). In particular, since each $s(\in\widetilde{s})$ is stable, by (MS-τ-2,7), we have $i_Y\leq|\widetilde{X}|$ with $Y\in\widetilde{Y}$ and

$$p_{i_Y}\overset{\tau}{\Longrightarrow}|p'''_Y \text{ for each } Y\in\widetilde{Y} \text{ and } p'\equiv C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p'''_Y}/\widetilde{Y}\}\notin F.$$

Then, by Lemma 5.5, it follows from (MS-τ-1) that, for each $Y\in\widetilde{Y}$, $p'''_Y\notin F$ and hence $p_{i_Y}\overset{\tau}{\Longrightarrow}_F|p'''_Y$ by Lemma 4.2. Further, since $\widetilde{p}\bowtie\widetilde{q}$ and $\widetilde{p}\sqsubseteq_{RS}\widetilde{q}$, there exist q'''_Y with $Y\in\widetilde{Y}$ such that

$$q_{i_Y}\overset{\tau}{\Longrightarrow}_F|q'''_Y \text{ and } p'''_Y\sqsubset_{\sim RS} q'''_Y.$$

On the other hand, since Z is active and occurs at most once in $C_{\widetilde{X},\widetilde{Z}}$ for each $Z\in\widetilde{Z}$, by Lemma 5.4, it follows from $\widetilde{r}\overset{\epsilon}{\Longrightarrow}\widetilde{t}$ that

$$C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\overset{\epsilon}{\Longrightarrow}C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}. \tag{6.6.1}$$

Moreover, by (MS-τ-3-ii), it follows from $q_{i_Y}\overset{\tau}{\Longrightarrow}_F|q'''_Y$ with $Y\in\widetilde{Y}$ that

$$C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}\overset{\epsilon}{\Longrightarrow}C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q'''_Y}/\widetilde{Y}\}. \tag{6.6.2}$$

Since $\widetilde{p}\bowtie\widetilde{q}$, $\widetilde{s}\sqsubset_{\sim RS}\widetilde{t}$ and $\widetilde{p'''_Y}\sqsubset_{\sim RS}\widetilde{q'''_Y}$, by $p'\equiv C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p'''_Y}/\widetilde{Y}\}\overset{\tau}{\not\longrightarrow}$ and Lemma 5.6, we can make the conclusion that $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q'''_Y}/\widetilde{Y}\}$ is stable. Hence $\mathcal{T}$ has a proper subtree with the root labelled with $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q'''_Y}/\widetilde{Y}\}F$ by (6.6.1) and (6.6.2), moreover, we also have $p'\equiv C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p'''_Y}/\widetilde{Y}\}\notin F$ and $\widetilde{p'''_Y}\sqsubset_{\sim RS}\widetilde{q'''_Y}$. Consequently, $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, $\widetilde{p'''_Y}$ and $\widetilde{q'''_Y}$ are what we seek. □

Lemma 6.7. For any $C_{\widetilde{X}}$ and processes $\widetilde{r}$ and $\widetilde{s}$, if $\widetilde{r}\bowtie\widetilde{s}$ and $\widetilde{r}\sqsubseteq_{RS}\widetilde{s}$, then $C_{\widetilde{X}}\{\widetilde{r}/\widetilde{X}\}\notin F$ implies $C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\}\notin F$.

Proof. Set

$$\Omega=\{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}:\widetilde{p}\bowtie\widetilde{q},\ \widetilde{p}\sqsubseteq_{RS}\widetilde{q},\ B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\notin F \text{ and } B_X \text{ is a context}\}.$$

Let $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}\in\Omega$ and $\mathcal{T}$ be any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R},M_{\mathrm{CLL}_R})\vdash C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F$. Similar to Lemma 6.3, it suffices to show that $\mathcal{T}$ has a proper subtree with the root labelled with sF for some $s\in\Omega$. We distinguish six cases based on the form of $C_{\widetilde{X}}$.

Case 1 $C_{\widetilde{X}}$ is closed or $C_{\widetilde{X}}\equiv X_i$.

In such situation, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}\notin F$ because of $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}\notin F$ and $\widetilde{p}\sqsubseteq_{RS}\widetilde{q}$. Thus there is no proof tree of $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F$. Hence the conclusion holds trivially.

Case 2 $C_{\widetilde{X}} \equiv \alpha.B_{\widetilde{X}}$.

Then the last rule applied in $\mathcal{T}$ is $\frac{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}{\alpha.B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$. Moreover $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$. Hence $B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$, as desired.

Case 3 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \vee D_{\widetilde{X}}$.

Obviously, the last rule applied in $\mathcal{T}$ is $\frac{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F, D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \vee D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$. Due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we have either $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ or $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, which implies $B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$ or $D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$. Thus $\mathcal{T}$ contains a proper subtree with the root labelled with sF for some $s \in \Omega$.

Case 4 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \odot D_{\widetilde{X}}$ with $\odot \in \{\Box, \|_A\}$.

W.l.o.g, assume the last rule applied in $\mathcal{T}$ is $\frac{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \odot D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$. Since $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we get $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, which implies $B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$, as desired.

Case 5 $C_{\widetilde{X}} \equiv \langle Y|E\rangle$.

Clearly, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{\widetilde{q}/\widetilde{X}\}F}{\langle Y|E\rangle\{\widetilde{q}/\widetilde{X}\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{\widetilde{q}/\widetilde{X}\}F}.$$

For the first alternative, we have $\langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \notin F$ because of $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, and hence $\langle t_Y|E\rangle\{\widetilde{q}/\widetilde{X}\} \in \Omega$.

For the second alternative, due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we get

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow}_F |s \text{ for some } s.$$

For such transition, by Lemma 5.17, there exist $C'_{\widetilde{X},\widetilde{Z}}$, $i_Z \leq |\widetilde{X}|$ and p'_Z with $Z \in \widetilde{Z}$ that realize (MS-τ-1) – (MS-τ-7). Amongst them, by (MS-τ-2,7), we have

$$p_{i_Z} \overset{\tau}{\Longrightarrow} |p'_Z \text{ with } Z \in \widetilde{Z} \text{ and } s \equiv C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Z}/\widetilde{Z}\} \notin F. \tag{6.7.1}$$

Thus, for each $Z \in \widetilde{Z}$, by (MS-τ-1) and Lemma 5.5, it follows that $p'_Z \notin F$, and hence $p_{i_Z} \overset{\tau}{\Longrightarrow}_F |p'_Z$ by Lemma 4.2. Further, since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that for each $Z \in \widetilde{Z}$,

$$q_{i_Z} \overset{\tau}{\Longrightarrow}_F |q'_Z \text{ and } p'_Z \sqsubset_{\sim RS} q'_Z \text{ for some } q'_Z. \tag{6.7.2}$$

Then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\}$ by (MS-τ-3-ii). On the other hand, since $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p'_Z} \sqsubset_{\sim RS} \widetilde{q'_Z}$, by Lemma 5.6, it follows from $s \equiv C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Z}/\widetilde{Z}\} \overset{\tau}{\not\longrightarrow}$ that $C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\}$ is stable. Therefore

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\}.$$

Hence $\mathcal{T}$ has a proper subtree with the root labelled with $C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\}F$, moreover

$C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Z}/\widetilde{Z}\} \in \Omega$ due to (6.7.1) and (6.7.2).

Case 6 $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \wedge D_{\widetilde{X}}$.

The argument splits into five subcases depending on the last rule applied in $\mathcal{T}$.

Case 6.1 $\frac{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$.

Similar to Case 4, omitted.

Case 6.2 $\frac{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} r,\, D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \not\xrightarrow{a},\, C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \not\xrightarrow{\tau}}{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$.

Clearly, $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ and $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$. Moreover, by $\widetilde{p} \bowtie \widetilde{q}$ and Lemma 5.6, we have $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \not\xrightarrow{\tau}$ because of $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \not\xrightarrow{\tau}$. Further, by Lemma 6.2, we also have $B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a}$ and $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \not\xrightarrow{a}$. So, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \wedge D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \in F$ by the rule Rp_{10}, which contradicts $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$. Hence such case is impossible.

Case 6.3 $\frac{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} r',\, \{rF : C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} r\}}{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$.

It follows from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ that

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ for some } s. \tag{6.7.3}$$

Since $\widetilde{p} \bowtie \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau}$, by Lemma 5.6, we get $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau}$. Then, by (6.7.3), we have

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau}_F t \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ for some } t.$$

For the τ-labelled transition leading to t in the above, either the clause (1) or (2) in Lemma 5.6 holds.

For the former, there exists $C'_{\widetilde{X}}$ such that $t \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$. Hence $C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F$ is one of premises of the last inferring step in $\mathcal{T}$. Moreover, it is evident that $C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$.

For the latter, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i_0 \leq |\widetilde{X}|$ that realize (P-τ-1) – (P-τ-4). In particular, by (P-τ-2), we have

$$t \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \text{ for some } p' \text{ with } p_{i_0} \xrightarrow{\tau} p'.$$

Further, since $t \stackrel{\epsilon}{\Longrightarrow}_F |s$ and Z is 1-active in $C''_{\widetilde{X},Z}$, by Lemma 5.21 and 4.2, there exists p'' such that $p' \stackrel{\epsilon}{\Longrightarrow} |p''$ and

$$t \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \stackrel{\epsilon}{\Longrightarrow}_F C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p''/Z\} \stackrel{\epsilon}{\Longrightarrow}_F |s.$$

Moreover, $p'' \notin F$ by Lemma 5.5. Hence $p_{i_0} \xrightarrow{\tau}_F p' \stackrel{\epsilon}{\Longrightarrow}_F |p''$ by Lemma 4.2. Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that

$$q_{i_0} \xrightarrow{\tau}_F q' \stackrel{\epsilon}{\Longrightarrow}_F |q'' \text{ and } p'' \underset{\sim}{\sqsubset}_{RS} q'' \text{ for some } q' \text{ and } q''. \tag{6.7.4}$$

Then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}$ by (P-τ-4). Therefore, $\mathcal{T}$ contains a proper subtree

$\mathcal{T}'$ with the root labelled with $C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}F$. In order to complete the proof, it is enough to show that $\mathcal{T}'$ contains a node labelled with $s'F$ for some $s' \in \Omega$. Since Z is 1-active, $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, $\widetilde{p} \bowtie \widetilde{q}$, $q' \overset{\epsilon}{\Longrightarrow} |q''$, $p'' \underset{\sim}{\sqsubset}_{RS} q''$ and $C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p''/Z\} \notin F$, by Lemma 6.6, there exist $C^*_{\widetilde{X},Z,\widetilde{Y}}$ and q'''_Y, p'''_Y with $Y \in \widetilde{Y}$ such that

(a.1) $\mathcal{T}'$ has a subtree with the root labelled with $C^*_{\widetilde{X},Z,\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, q''/Z, \widetilde{q'''_Y}/\widetilde{Y}\}F$,

(a.2) $C^*_{\widetilde{X},Z,\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, p''/Z, \widetilde{p'''_Y}/\widetilde{Y}\} \notin F$, and

(a.3) $\widetilde{p'''_Y} \underset{\sim}{\sqsubset}_{RS} \widetilde{q'''_Y}$.

Clearly, $C^*_{\widetilde{X},Z,\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, q''/Z, \widetilde{q'''_Y}/\widetilde{Y}\} \in \Omega$ due to (a.2), (a.3) and (6.7.4), as desired.

Case 6.4 $\frac{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} r', \{rF : C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} r\}}{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F} (a \in Act)$.

Since $\widetilde{p} \bowtie \widetilde{q}$, by Lemma 5.6, it follows from $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a}$ that $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ is stable. Further, since $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we get $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a}$ by Lemma 6.2. So, by Theorem 4.2 and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we have

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a}_F t \overset{\epsilon}{\Longrightarrow}_F |s \text{ for some } t \text{ and } s. \tag{6.7.5}$$

For the a-labelled transition involving in (6.7.5), by Lemma 5.9, there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ that satisfy (CP-a-1) – (CP-a-4). In particular, by (CP-a-3-ii), there exist $i_Y \leq |\widetilde{X}|$ and p'_Y with $Y \in \widetilde{Y}$ such that

$$p_{i_Y} \xrightarrow{a} p'_Y \text{ and } t \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}.$$

Moreover, by (CP-a-1) and (CP-a-3-i), we have

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}.$$

Hence $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\} \notin F$ by $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ and Lemma 5.15. Further, for each $Y \in \widetilde{Y}$, since Y is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ (i.e., (CP-a-2)), by Lemma 5.5, $p_{i_Y} \notin F$.

On the other hand, for the transition $t \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow}_F |s$ in (6.7.5), by Lemma 5.21, it follows from each $Y (\in \widetilde{Y})$ is 1-active in $C''_{\widetilde{X},\widetilde{Y}}$ (i.e., (CP-a-2)) that there exist p''_Y with $Y \in \widetilde{Y}$ such that $p'_Y \overset{\epsilon}{\Longrightarrow} |p''_Y$ and

$$t \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} |s.$$

Since $s \notin F$, we obtain $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \notin F$ by Lemma 4.2, which implies that $p''_Y \notin F$ with $Y \in \widetilde{Y}$ due to Lemma 5.5. Thus

$$p_{i_Y} \xrightarrow{a}_F p'_Y \overset{\epsilon}{\Longrightarrow}_F |p''_Y \text{ for each } Y \in \widetilde{Y}.$$

Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that, for each $Y \in \widetilde{Y}$, there exist q'_Y and q''_Y such that

$$q_{i_Y} \xrightarrow{a}_F q'_Y \overset{\epsilon}{\Longrightarrow}_F |q''_Y \text{ and } p''_Y \underset{\sim}{\sqsubset}_{RS} q''_Y. \tag{6.7.6}$$

Then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\}$ by (CP-a-3-iii). Hence $\mathcal{T}$ has a proper subtree $\mathcal{T}'$ with the root labelled with $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\}F$. In order to complete the proof, it suffices to show that $\mathcal{T}'$ contains a node labelled with $s'F$ for some $s' \in \Omega$. Since each $Y(\in \widetilde{Y})$ is 1-active in $C''_{\widetilde{X},\widetilde{Y}}$, $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, $\widetilde{p} \bowtie \widetilde{q}$, $\widetilde{q'_Y} \stackrel{\epsilon}{\Longrightarrow} |\widetilde{q''_Y}$, $\widetilde{p''_Y} \underset{\sim RS}{\sqsubset} \widetilde{q''_Y}$ and $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p''_Y}/\widetilde{Y}\} \notin F$, by Lemma 6.6, there exist $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}$ and q'''_Z, p'''_Z with $Z \in \widetilde{Z}$ such that

(b.1) $\mathcal{T}'$ has a subtree with the root labelled with $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{q''_Y}/\widetilde{Y},\widetilde{q'''_Z}/\widetilde{Z}\}F$,

(b.2) $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{p''_Y}/\widetilde{Y},\widetilde{p'''_Z}/\widetilde{Z}\} \notin F$, and

(b.3) $\widetilde{p'''_Z} \underset{\sim RS}{\sqsubset} \widetilde{q'''_Z}$.

Obviously, $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{q''_Y}/\widetilde{Y},\widetilde{q'''_Z}/\widetilde{Z}\} \in \Omega$ due to (b.2), (b.3) and (6.7.6), as desired.

Case 6.5 $\frac{\{rF : B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}\wedge D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}\stackrel{\epsilon}{\Longrightarrow}|r\}}{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}\wedge D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$.

Similar to the second alternative in Case 5, omitted. □

In the remainder of this section, we shall prove that $\sqsubseteq_{RS}$ indeed is precongruent. Let us first recall a distinct but equivalent formulation of $\sqsubseteq_{RS}$ due to Van Glabbeek (Lüttgen and Vogler 2010).

Definition 6.2. A relation $\mathcal{R} \subseteq T(\Sigma_{\mathrm{CLL}_R}) \times T(\Sigma_{\mathrm{CLL}_R})$ is said to be an alternative ready simulation relation, if for any $(p,q) \in \mathcal{R}$ and $a \in Act$

(RSi) $p \stackrel{\epsilon}{\Longrightarrow}_F |p'$ implies $\exists q'.q \stackrel{\epsilon}{\Longrightarrow}_F |q'$ and $(p',q') \in \mathcal{R}$;

(RSiii) $p \stackrel{a}{\Longrightarrow}_F |p'$ and p,q stable implies $\exists q'.q \stackrel{a}{\Longrightarrow}_F |q'$ and $(p',q') \in \mathcal{R}$;

(RSiv) $p \notin F$ and p,q stable implies $\mathcal{I}(p) = \mathcal{I}(q)$.

We write $p \sqsubseteq_{ALT} q$ if there exists an alternative ready simulation relation $\mathcal{R}$ with $(p,q) \in \mathcal{R}$.

The next proposition reveals that this definition agrees with the one given in Def. 2.3.

Proposition 6.1. $\sqsubseteq_{RS}=\sqsubseteq_{ALT}$.

Proof. See Prop. 13 in (Lüttgen and Vogler 2010). □

One advantage of Def. 6.2 lies in that, given p and q, we can prove $p \sqsubseteq_{RS} q$ by means of giving an alternative ready simulation relation relating them. It is well known that up-to technique is a tractable way for such coinduction proof. Here we introduce the notion of an alternative ready relation up to $\underset{\sim RS}{\sqsubset}$ as follows.

Definition 6.3 (ALT up to $\underset{\sim RS}{\sqsubset}$). A relation $\mathcal{R} \subseteq T(\Sigma_{\mathrm{CLL}_R}) \times T(\Sigma_{\mathrm{CLL}_R})$ is said to be an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$, if for any $(p,q) \in \mathcal{R}$ and $a \in Act$

(ALT-upto-1) $p \stackrel{\epsilon}{\Longrightarrow}_F |p'$ implies $\exists q'.q \stackrel{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$;

(ALT-upto-2) $p \stackrel{a}{\Longrightarrow}_F |p'$ and p,q stable implies $\exists q'.q \stackrel{a}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$;

(ALT-upto-3) $p \notin F$ and p,q stable implies $\mathcal{I}(p) = \mathcal{I}(q)$.

As usual, given a relation $\mathcal{R}$ satisfying the conditions (ALT-upto-1,2,3), in general, $\mathcal{R}$ in itself is not an alternative ready simulation relation. But simple result below ensures that up-to technique based on the above notion is sound.

Lemma 6.8. If a relation $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim}{\sqsubset}_{RS}$ then $\mathcal{R} \subseteq \sqsubseteq_{RS}$.

Proof. By Prop. 6.1, it is enough to prove that the relation $\sqsubseteq_{RS} \circ \mathcal{R} \circ \sqsubseteq_{RS}$ is an alternative ready simulation. We leave it to the reader. □

Now we are ready to prove the main result of this section: $\sqsubseteq_{RS}$ is precongruent w.r.t all operations in CLL_R. We shall divide the proof into the next two lemmas.

Lemma 6.9. $C_X\{p/X\} =_{RS} C_X\{\tau.p/X\}$ for any context C_X and stable process p.

Proof. Let p be any stable process. First, we shall show that $C_X\{p/X\} \sqsubseteq_{RS} C_X\{\tau.p/X\}$. Set

$$\mathcal{R} \triangleq \{(B_X\{p/X\}, B_X\{\tau.p/X\}) : B_X \text{ is a context}\}.$$

By Prop. 6.1 and Lemma 6.8, it is enough to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim}{\sqsubset}_{RS}$. Let $(C_X\{p/X\}, C_X\{\tau.p/X\}) \in \mathcal{R}$.

(ALT-upto-1) Assume that $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |p'$. For this transition, since p is stable, by Lemma 5.17, there exists a stable context C'_X such that

$$p' \equiv C'_X\{p/X\} \text{ and } C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_X\{\tau.p/X\}. \tag{6.9.1}$$

Moreover, by Lemma 5.18, it follows from $\tau.p \stackrel{\tau}{\longrightarrow} |p$ that

$$C'_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |r \text{ for some } r. \tag{6.9.2}$$

For this transition, by Lemma 5.17, there exists a context $C''_{X,\widetilde{Y}}$ with $X \notin \widetilde{Y}$ such that $r \equiv C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ and

$$p' \equiv C'_X\{p/X\} \Rrightarrow C''_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}. \tag{6.9.3}$$

Since $p' \notin F$, by Lemma 5.15, we get $C''_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \notin F$. Further, by Lemma 6.3, $r \equiv C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \notin F$. So, by (6.9.1), (6.9.2) and Lemma 4.2, we obtain

$$C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}.$$

Moreover, by Lemma 5.16, it follows from (6.9.3) that

$$p' \underset{\sim}{\sqsubset}_{RS} C''_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \mathcal{R} C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}.$$

(ALT-upto-2) Assume that $C_X\{p/X\}$ and $C_X\{\tau.p/X\}$ are stable and $C_X\{p/X\} \stackrel{a}{\Longrightarrow}_F |p'$. Hence $C_X\{p/X\} \stackrel{a}{\longrightarrow}_F r \stackrel{\epsilon}{\Longrightarrow}_F |p'$ for some r. Moreover, by Lemma 6.3 and $C_X\{p/X\} \notin F$, we have

$$C_X\{\tau.p/X\} \notin F. \tag{6.9.4}$$

On the other hand, for the a-labelled transition $C_X\{p/X\} \stackrel{a}{\longrightarrow}_F r$, by Lemma 5.9, there

exist C'_X, $C'_{X,\widetilde{Y}}$ and $C''_{X,\widetilde{Y}}$ that realize (CP-a-1) – (CP-a-4). By (CP-a-1) and (CP-a-3-i), we have

$$C_X\{\tau.p/X\} \Rrightarrow C'_X\{\tau.p/X\} \equiv C'_{X,\widetilde{Y}}\{\tau.p/X, \tau.p/\widetilde{Y}\}.$$

If $\widetilde{Y} \neq \emptyset$ then, by (CP-a-2) and Lemma 5.4, we have $C'_{X,\widetilde{Y}}\{\tau.p/X, \tau.p/\widetilde{Y}\} \xrightarrow{\tau}$, and hence $C_X\{\tau.p/X\} \xrightarrow{\tau}$ by Lemma 5.12, which contradicts that $C_X\{\tau.p/X\}$ is stable. Thus $\widetilde{Y} = \emptyset$. So, $r \equiv C''_{X,\widetilde{Y}}\{p/X\}$ by (CP-a-3-ii) and

$$C_X\{\tau.p/X\} \xrightarrow{a} C''_{X,\widetilde{Y}}\{\tau.p/X\} \text{ by (CP-}a\text{-3-iii) and } C_X\{\tau.p/X\} \not\xrightarrow{\tau}. \tag{6.9.5}$$

Moreover, by (ALT-upto-1), it follows from $(C''_{X,\widetilde{Y}}\{p/X\}, C''_{X,\widetilde{Y}}\{\tau.p/X\}) \in \mathcal{R}$ and $r \equiv C''_{X,\widetilde{Y}}\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$ that $C''_{X,\widetilde{Y}}\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$ for some q'. Moreover, we also have $C_X\{\tau.p/X\} \overset{a}{\Longrightarrow}_F |q'$ due to (6.9.4) and (6.9.5), as desired.

(ALT-upto-3) Immediately follows from Lemma 6.2.

Next we intend to prove $C_X\{\tau.p/X\} \sqsubseteq_{RS} C_X\{p/X\}$. Set

$$\mathcal{R} \triangleq \{(B_X\{\tau.p/X\}, B_X\{p/X\}) : B_X \text{ is a context}\}.$$

Similarly, it is enough to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$. Let $(C_X\{\tau.p/X\}, C_X\{p/X\}) \in \mathcal{R}$. In the following, we will check that the condition (ALT-upto-1) holds. For (ALT-upto-2) and (ALT-upto-3), the proof are analogous to the preceding one and omitted.

Assume that $C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$. For this transition, by Lemma 5.20, there exist r and stable context C^*_X such that $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{p/X\}$ and

$$C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} |r \Rrightarrow p'. \tag{6.9.6}$$

Moreover, since p is stable, so is $C^*_X\{p/X\}$ by Lemma 5.6. Due to $r \Rrightarrow p'$ and $p' \notin F$, by Lemma 5.15, we get $r \notin F$. Hence $C^*_X\{\tau.p/X\} \notin F$ by (6.9.6) and Lemma 4.2. Then $C^*_X\{p/X\} \notin F$ by Lemma 6.5. Thus

$$C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |C^*_X\{p/X\}.$$

To complete the proof, it remains to prove that $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} C^*_X\{p/X\}$. For the transition $C^*_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} |r$ in (6.9.6), by Lemma 5.17, there exists a stable context $C'^*_{X,\widetilde{Y}}$ such that $r \equiv C'^*_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \Rrightarrow p'$ and $C^*_X\{p/X\} \Rrightarrow C'^*_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}$, which, by Lemma 5.16, implies

$$p' \underset{\sim RS}{\sqsubset} C'^*_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}\mathcal{R}C'^*_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \underset{\sim RS}{\sqsubset} C^*_X\{p/X\}.$$

□

Lemma 6.10. If $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \sqsubseteq_{RS} C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any $C_{\widetilde{X}}$.

Proof. Set

$$\mathcal{R} \triangleq \{(B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}, B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}) : \widetilde{p} \bowtie \widetilde{q}, \widetilde{p} \sqsubseteq_{RS} \widetilde{q} \text{ and } B_{\widetilde{X}} \text{ is a context}\}.$$

Similarly, it suffices to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim}{\sqsubset}_{RS}$. Suppose $(C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}, C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}) \in \mathcal{R}$. Then, by Lemma 6.2, it is obvious that such pair satisfies the condition (ALT-upto-3). In the following, we consider two remaining conditions in turn.

(ALT-upto-1) Assume that $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow}_F |s$. For this transition, by Lemma 5.17, there exist $C'_{\widetilde{X},\widetilde{Y}}$, $i_Y \leq |\widetilde{X}|$ and p'_Y with $Y \in \widetilde{Y}$ that satisfy (MS-τ-1) – (MS-τ-7). In particular, by (MS-τ-2,7), we have

$$p_{i_Y} \overset{\tau}{\Longrightarrow} |p'_Y \text{ and } s \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \notin F.$$

Then, by (MS-τ-1) and Lemma 5.5, $p'_Y \notin F$ and hence $p_{i_Y} \overset{\tau}{\Longrightarrow}_F |p'_Y$ by Lemma 4.2 for each $Y \in \widetilde{Y}$. Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that there exist q'_Y with $Y \in \widetilde{Y}$ such that

$$q_{i_Y} \overset{\tau}{\Longrightarrow}_F |q'_Y \text{ and } p'_Y \underset{\sim}{\sqsubset}_{RS} q'_Y. \tag{6.10.1}$$

So, by (MS-τ-3-ii), we get

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}.$$

Moreover, by Lemma 5.6, it follows from $s \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\tau}{\not\longrightarrow}$, $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p'_Y} \underset{\sim}{\sqsubset}_{RS} \widetilde{q'_Y}$ that

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \overset{\tau}{\not\longrightarrow}.$$

On the other hand, by Lemma 6.7 and $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \notin F$, we get $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \notin F$. Hence, by Lemma 4.2, we obtain

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow}_F |C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}.$$

Moreover, $(C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}, C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}) \in \underset{\sim}{\sqsubset}_{RS} \mathcal{R} \underset{\sim}{\sqsubset}_{RS}$ due to (6.10.1) and the reflexivity of $\underset{\sim}{\sqsubset}_{RS}$.

(ALT-upto-2) Let $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ be stable and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{a}{\Longrightarrow}_F |s$. Then

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{a}{\longrightarrow}_F r \overset{\epsilon}{\Longrightarrow}_F |s \text{ for some } r. \tag{6.10.2}$$

Moreover, by Lemma 6.7, it follows from $\widetilde{p} \bowtie \widetilde{q}$, $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ that

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \notin F. \tag{6.10.3}$$

For the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{a}{\longrightarrow} r$, by Lemma 5.9, there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ that satisfy (CP-a-1) – (CP-a-4). In particular, by (CP-a-3-ii), there exist $i_Y \leq |\widetilde{X}|$ and p'_Y with $Y \in \widetilde{Y}$ such that $p_{i_Y} \overset{a}{\longrightarrow} p'_Y$ and $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$. Moreover, by (CP-$a$-1) and (CP-$a$-3-i), we have

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}.$$

Hence $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\} \notin F$ by $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ and Lemma 5.15. Further, since each

$Y(\in \widetilde{Y})$ is 1-active in $C'_{X,\widetilde{Y}}$, by Lemma 5.5, we get

$$p_{i_Y} \notin F \text{ with } Y \in \widetilde{Y}. \tag{6.10.4}$$

By Lemma 5.21, it follows from $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} |s$ in (6.10.2) that for each $Y \in \widetilde{Y}$, there exists p''_Y such that $p'_Y \overset{\epsilon}{\Longrightarrow} |p''_Y$ and

$$C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} |s.$$

Then $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \notin F$ due to $s \notin F$ and Lemma 4.2, and hence $p''_Y \notin F$ with $Y \in \widetilde{Y}$ by Lemma 5.5. Therefore, by (6.10.4) and Lemma 4.2, we have

$$p_{i_Y} \overset{a}{\longrightarrow}_F p'_Y \overset{\epsilon}{\Longrightarrow}_F |p''_Y \text{ for each } Y \in \widetilde{Y}.$$

On the other hand, since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that for each $Y \in \widetilde{Y}$, there exist q'_Y and q''_Y such that $q_{i_Y} \overset{a}{\longrightarrow}_F q'_Y \overset{\epsilon}{\Longrightarrow}_F |q''_Y$ and $p''_Y \underset{\sim RS}{\sqsubset} q''_Y$. By (CP-$a$-3-iii), we get

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{a}{\longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}. \tag{6.10.5}$$

Further, by Lemma 5.4 and (CP-a-2), we obtain

$$C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}\}. \tag{6.10.6}$$

Clearly, $(C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\}, C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}\}) \in \mathcal{R}$. So, by $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow}_F |s$ and (ALT-upto-1), there exists t such that $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow}_F |t$ and $s \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} t$, moreover, we also have $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{a}{\Longrightarrow}_F |t$ due to (6.10.3), (6.10.5), (6.10.6) and Lemma 4.2, as desired. □

We are now in a position to state the main result of this section.

Theorem 6.1 (Precongruence). If $p \sqsubseteq_{RS} q$ then $C_X\{p/X\} \sqsubseteq_{RS} C_X\{q/X\}$ for any context C_X.

Proof. By Lemma 6.9 and 6.10, it immediately follows from $\tau.p =_{RS} p \sqsubseteq_{RS} q =_{RS} \tau.q$. □

As an immediate consequence of this theorem, we also have

Corollary 6.1. If $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \sqsubseteq_{RS} C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any context $C_{\widetilde{X}}$.

Proof. Applying Theorem 6.1 finitely many times. □

7. Unique solution of equations

This section focuses on the solutions of equations. Especially, we shall prove that the equation $X =_{RS} t_X$ has at most one consistent solution modulo $=_{RS}$ provided that X is strongly guarded and does not occur in any scope of conjunctions in t_X, moreover, the process $\langle X|X = t_X\rangle$ indeed is the unique consistent solution whenever such equation has a consistent solution. We begin with giving two results on the inconsistency predicate F.

Lemma 7.1. For any stable processes $p, q \notin F$ and context C_X such that X does not occur in any scope of conjunctions, if $C_X\{p/X\} \in F$ then $C_X\{q/X\} \in F$.

Proof. Assume that $C_X\{p/X\} \in F$ and $\mathcal{T}$ is any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{p/X\}F$. We proceed by induction on the depth of $\mathcal{T}$. The argument is a routine case analysis on C_X. Moreover, since X does not occur in any scope of conjunctions, in addition to that C_X is closed, the form of C_X is one of the following: X, $\alpha.B_X$, $B_X \odot D_X$ with $\odot \in \{\vee, \Box, \|_A\}$ and $\langle Y|E\rangle$. Here, we give the proof only for the case $C_X \equiv \langle Y|E\rangle$, the other cases are straightforward and omitted.

For the case $C_X \equiv \langle Y|E\rangle$, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{p/X\}F}{\langle Y|E\rangle\{p/X\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{p/X\}F}.$$

For the first alternative, we have $\langle t_Y|E\rangle\{q/X\} \in F$ by IH, and hence $C_X\{q/X\} \equiv \langle Y|E\rangle\{q/X\} \in F$.

For the second alternative, assume $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$. Since q is stable, by Lemma 5.17, $s \equiv C'_X\{q/X\}$ for some stable C'_X such that X does not occur in any scope of conjunctions in C'_X and $\langle Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_X\{p/X\}$. Moreover, since p is stable, so is $C'_X\{p/X\}$. Thus there exists a proper subtree of $\mathcal{T}$ with the root labelled with $C'_X\{p/X\}F$. So, by IH, $s \equiv C'_X\{q/X\} \in F$. Hence $C_X\{q/X\} \in F$ by Theorem 4.2, as desired. $\Box$

This result is of independent interest, but its principal use is that it will serve as an important step in demonstrating the next lemma, which reveals that the above result still holds if it is deleted from the hypotheses that q and p are stable.

Lemma 7.2. For any processes $p, q \notin F$ and context C_X such that X does not occur in any scope of conjunctions, if $C_X\{p/X\} \in F$ then $C_X\{q/X\} \in F$.

Proof. Suppose that $C_X\{p/X\} \in F$. We proceed by induction on the depth of the proof tree $\mathcal{T}$ of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{p/X\}F$. Similar to the preceding lemma, we handle only the case $C_X \equiv \langle Y|E\rangle$. In such situation, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{p/X\}F}{\langle Y|E\rangle\{p/X\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{p/X\}F}.$$

The argument for the former is same to one in Lemma 7.1 and omitted. In the following, we consider the latter and suppose $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$. By Theorem 4.2, it is not difficult to see that, to complete the proof, it suffices to prove that $s \in F$. By Lemma 5.20, there exist t and stable context C^*_X such that

$$\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |t \Rrightarrow s$$

and

$$\langle Y|E\rangle\{r/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{r/X\} \text{ for any } r. \tag{7.2.1}$$

In particular, we have $\langle Y|E\rangle\{a_X.0/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{a_X.0/X\}$ where a_X is a fresh visible action. For this transition, applying Lemma 5.6 finitely times (notice that, in this procedure, since $a_X.0$ is stable, the clause (2) in Lemma 5.6 is always false), then by the

clause (1) in Lemma 5.6, we get the sequence below

$$\langle Y|E\rangle\{a_X.0/X\} \equiv C_X^0\{a_X.0/X\} \xrightarrow{\tau} C_X^1\{a_X.0/X\} \xrightarrow{\tau} \cdots \xrightarrow{\tau} C_X^n\{a_X.0/X\} \equiv C_X^*\{a_X.0/X\}.$$

Here $n \geq 0$ and for each $1 \leq i \leq n$, C_X^i satisfies (C-τ-1,2,3) in Lemma 5.6. Since X does not occur in any scope of conjunctions in $\langle Y|E\rangle$, by (C-τ-3-iv), neither does X in C_X^n. On the other hand, by Lemma 5.19, we have $C_X^n \equiv C_X^*$. Hence X does not occur in any scope of conjunctions in C_X^*.

If p is stable then so is $C_X^*\{p/X\}$ by Lemma 5.6. Thus, by (7.2.1), $C_X^*\{p/X\}F$ is one of premises in the last inferring step in $\mathcal{T}$. Hence $C_X^*\{q/X\} \in F$ by applying IH. Then $t \in F$ by Lemma 4.2. Further, by Lemma 5.15, it follows from $t \Rrightarrow s$ that $s \in F$, as desired.

Next we consider another case where p is not stable. In such situation, due to $p \notin F$, we have

$$p \overset{\tau}{\Longrightarrow}_F |p^* \text{ for some } p^*. \tag{7.2.2}$$

In the following, we distinguish two cases based on whether q is stable.

Case 1 q is stable.

Then, for the transition $\langle Y|E\rangle\{q/X\} \overset{\epsilon}{\Longrightarrow} |s$, by Lemma 5.17, we have $s \equiv C'_X\{q/X\}$ for some stable C'_X such that X does not occur in any scope of conjunctions and $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C'_X\{p/X\}$. Moreover, by Lemma 5.18, it follows from (7.2.2) that

$$C'_X\{p/X\} \overset{\epsilon}{\Longrightarrow} |p' \text{ for some } p'.$$

For this transition, by Lemma 5.17, there exist a stable context $C'^*_{X,\widetilde{Y}}$ and stable processes p'_Y with $Y \in \widetilde{Y}$ that realize (MS-τ-1) – (MS-τ-7). In particular, by (MS-τ-3-ii), it follows from (7.2.2) that

$$C'_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C'^*_{X,\widetilde{Y}}\{p/X, p^*/\widetilde{Y}\}.$$

Moreover, by (MS-τ-2), $p' \equiv C'^*_{X,\widetilde{Y}}\{p/X, \widetilde{p'_Y}/\widetilde{Y}\}$. Then, since $p' \equiv C'^*_{X,\widetilde{Y}}\{p/X, \widetilde{p'_Y}/\widetilde{Y}\}$ and p^* are stable, by Lemma 5.6, so is $C'^*_{X,\widetilde{Y}}\{p/X, p^*/\widetilde{Y}\}$. Thus, $C'^*_{X,\widetilde{Y}}\{p/X, p^*/\widetilde{Y}\}F$ is one of premises of the last inferring step in $\mathcal{T}$. Then, by (MS-τ-6) and IH, we obtain

$$C'^*_{X,\widetilde{Y}}\{q/X, p^*/\widetilde{Y}\} \in F.$$

Further, by (MS-τ-6) and Lemma 7.1, we get

$$C'^*_{X,\widetilde{Y}}\{q/X, q/\widetilde{Y}\} \in F.$$

On the other hand, due to the stableness of C'_X, by (MS-τ-4), we have

$$C'_X\{q/X\} \Rrightarrow C'^*_{X,\widetilde{Y}}\{q/X, q/\widetilde{Y}\}.$$

Hence $s \equiv C'_X\{q/X\} \in F$ by Lemma 5.15, as desired.

Case 2 q is not stable.

By Lemma 5.17, for the transition $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$, there exist a stable context $C'_{X,\widetilde{Z}}$ and q'_Z with $Z \in \widetilde{Z}$ that satisfy (MS-τ-1) – (MS-τ-7). Amongst them, by (MS-τ-2,7),

$$q \stackrel{\tau}{\Longrightarrow} |q'_Z \text{ with } Z \in \widetilde{Z} \text{ and } s \equiv C'_{X,\widetilde{Z}}\{q/X, \widetilde{q'_Z}/\widetilde{Z}\}. \tag{7.2.3}$$

If $q'_Z \in F$ for some $Z \in \widetilde{Z}$ then by Lemma 5.5, we get $s \in F$ (notice that each Z in $\widetilde{Z}$ is 1-active), as desired. In the following, we handle another case where

$$q'_Z \notin F \text{ for each } Z \in \widetilde{Z}. \tag{7.2.4}$$

By (MS-τ-3-ii), it follows from (7.2.2) that

$$C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_{X,\widetilde{Z}}\{p/X, p^*/\widetilde{Z}\}.$$

Since $p \stackrel{\tau}{\longrightarrow}$, $q \stackrel{\tau}{\longrightarrow}$, $p^* \stackrel{\tau}{\not\longrightarrow}$, $q'_Z \stackrel{\tau}{\not\longrightarrow}$ with $Z \in \widetilde{Z}$ and $s \equiv C'_{X,\widetilde{Z}}\{q/X, \widetilde{q'_Z}/\widetilde{Z}\} \stackrel{\tau}{\not\longrightarrow}$, by Lemma 5.6, $C'_{X,\widetilde{Z}}\{p/X, p^*/\widetilde{Z}\}$ is stable. Hence $\mathcal{T}$ has a proper subtree with the root labelled with $C'_{X,\widetilde{Z}}\{p/X, p^*/\widetilde{Z}\}F$. Then $C'_{X,\widetilde{Z}}\{q/X, p^*/\widetilde{Z}\} \in F$ by (MS-τ-6) and IH. Further, by Lemma 7.1, it follows from (7.2.3) and (7.2.4) that $s \equiv C'_{X,\widetilde{Z}}\{q/X, \widetilde{q'_Z}/\widetilde{Z}\} \in F$, as desired. $\square$

We shall use the notation $Dep(\mathcal{T})$ to denote the depth of a given proof tree $\mathcal{T}$. Given p, q and $\alpha \in Act_\tau$, for any proof tree $\mathcal{T}$ of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \stackrel{\alpha}{\longrightarrow} q$, it is evident that $\mathcal{T}$ involves only rules in Table 1. Moreover, since each rule in Table 1 has only finitely many premises, it is not difficult to show that $Dep(\mathcal{T}) < \omega$ by induction on the depth of $\mathcal{T}$. This makes it legitimate to use arithmetical expressions with the form like $\sum_{\mathcal{T}\in\Omega} Dep(\mathcal{T})$ where Ω is a finite set and each $\mathcal{T} \in \Omega$ is a proof tree for some labelled transition $p \stackrel{\alpha}{\longrightarrow} r$.

Definition 7.1. Given $p \stackrel{\epsilon}{\Longrightarrow}_F q$ and a finite set Ω of proof trees, we say that Ω is a *proof forest* for $p \stackrel{\epsilon}{\Longrightarrow}_F q$ if there exist p_i with $0 \le i \le n$ such that

(1) $p \equiv p_0 \stackrel{\tau}{\longrightarrow}_F p_1 \stackrel{\tau}{\longrightarrow}_F \cdots \stackrel{\tau}{\longrightarrow}_F p_n \equiv q$,
(2) for each $i < n$, Ω contains exactly one proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_i \stackrel{\tau}{\longrightarrow} p_{i+1}$, and
(3) for each $\mathcal{T} \in \Omega$, $\mathcal{T}$ is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_i \stackrel{\tau}{\longrightarrow} p_{i+1}$ for some $i < n$.

The depth of Ω is defined as $Dep(\Omega) \triangleq \sum_{\mathcal{T}\in\Omega} Dep(\mathcal{T})$. Similarly, we may define the notion of a proof forest for $p \stackrel{a}{\Longrightarrow}_F q$.

It is obvious that $p \stackrel{\epsilon}{\Longrightarrow}_F q$ (or, $p \stackrel{a}{\Longrightarrow}_F q$) holds if and only if there exists a proof forest for it. The following lemma will prove extremely useful in establishing the main result in this section and its proof involves the induction based on the depth of proof forests.

Lemma 7.3. Let C_X be any context where X is strongly guarded and does not occur in any scope of conjunctions. For any processes $p, q \notin F$ with $p \bowtie q$, if $p =_{RS} C_X\{p/X\}$ and $q =_{RS} C_X\{q/X\}$ then $p =_{RS} q$.

Proof. Suppose $p, q \notin F$ with $p \bowtie q$, $p =_{RS} C_X\{p/X\}$ and $q =_{RS} C_X\{q/X\}$. It is enough to prove that $p \sqsubseteq_{RS} q$. Put

$$\mathcal{R} \triangleq \{(B_X\{p/X\}, B_X\{q/X\}) : X \text{ does not occur in any scope of conjunctions in } B_X\}.$$

By Prop. 6.1 and Lemma 6.8, it suffices to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim}{\sqsubset}_{RS}$. Let $(B_X\{p/X\}, B_X\{q/X\}) \in \mathcal{R}$.

(**ALT-upto-1**) Assume that $B_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$ and Ω is any proof forest for it. Hence $B_X\{p/X\} \equiv p_0 \overset{\tau}{\longrightarrow}_F p_1 \overset{\tau}{\longrightarrow} \dots p_{n-1} \overset{\tau}{\longrightarrow}_F |p_n \equiv p'$ for some p_i with $0 \leq i \leq n$, and Ω exactly consists of proof trees $\mathcal{T}_i (0 \leq i < n)$ for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_i \overset{\tau}{\longrightarrow} p_{i+1}$. We intend to prove that there exists q' such that $B_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim}{\sqsubset}_{RS} \mathcal{R} \underset{\sim}{\sqsubset}_{RS} q'$ by induction on $Dep(\Omega)$. It is a routine case analysis on B_X. We treat only three cases as samples.

Case 1 $B_X \equiv X$.

Then $B_X\{p/X\} \equiv p \overset{\epsilon}{\Longrightarrow}_F |p'$. Thus it follows from $p =_{RS} C_X\{p/X\}$ that

$$C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |s \text{ and } p' \underset{\sim}{\sqsubset}_{RS} s \text{ for some } s.$$

Since X is strongly guarded and does not occur in any scope of conjunctions in C_X, by Lemma 5.17, there exists a stable context C'_X such that

(a.1) $s \equiv C'_X\{p/X\}$,
(a.2) X is strongly guarded and does not occur in any scope of conjunctions in C'_X, and
(a.3) $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow} C'_X\{q/X\}$.

Since $s \equiv C'_X\{p/X\} \overset{\tau}{\not\longrightarrow}$, by (a.2) and Lemma 5.10, we have $C'_X\{q/X\} \overset{\tau}{\not\longrightarrow}$. Moreover, by Lemma 7.2, $C'_X\{q/X\} \notin F$ follows from $C'_X\{p/X\} \notin F$ and $p, q \notin F$. Hence $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |C'_X\{q/X\}$ by (a.3) and Lemma 4.2. Further, it follows from $q =_{RS} C_X\{q/X\}$ that

$$q \overset{\epsilon}{\Longrightarrow}_F |q' \text{ and } C'_X\{q/X\} \underset{\sim}{\sqsubset}_{RS} q' \text{ for some } q'.$$

Therefore, $B_X\{q/X\} \equiv q \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim}{\sqsubset}_{RS} s \equiv C'_X\{p/X\} \mathcal{R} C'_X\{q/X\} \underset{\sim}{\sqsubset}_{RS} q'$.

Case 2 $B_X \equiv \langle Y|E\rangle$.

If $\langle Y|E\rangle\{p/X\}$ is stable then so is $\langle Y|E\rangle\{q/X\}$ by $p \bowtie q$ and Lemma 5.6. Moreover, by Lemma 7.2, we have $\langle Y|E\rangle\{q/X\} \notin F$ because of $\langle Y|E\rangle\{p/X\} \notin F$. Hence $\langle Y|E\rangle\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |\langle Y|E\rangle\{q/X\}$ and $(\langle Y|E\rangle\{p/X\}, \langle Y|E\rangle\{q/X\}) \in \underset{\sim}{\sqsubset}_{RS} \mathcal{R} \underset{\sim}{\sqsubset}_{RS}$ due to the reflexivity of $\underset{\sim}{\sqsubset}_{RS}$.

Next we handle another case where $\langle Y|E\rangle\{p/X\}$ is not stable. Clearly, the last rule applied in $\mathcal{T}_0$ is

$$\frac{\langle t_Y|E\rangle\{p/X\} \overset{\tau}{\longrightarrow} p_1}{\langle Y|E\rangle\{p/X\} \overset{\tau}{\longrightarrow} p_1} \text{ with } Y = t_Y \in E.$$

Thus, $\mathcal{T}_0$ contains a proper subtree, say $\mathcal{T}'_0$, which is a proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash \langle t_Y|E\rangle\{p/X\} \overset{\tau}{\longrightarrow} p_1$ and $Dep(\mathcal{T}'_0) < Dep(\mathcal{T}_0)$. Thus $\Omega' \triangleq \{\mathcal{T}'_0, \mathcal{T}_i : 1 \leq i \leq n-1\}$ is a

proof forest for $\langle t_Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |p'$, moreover

$$Dep(\Omega') < Dep(\Omega).$$

Then, by Lemma 5.2(5) and IH, we have $\langle t_Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'$ and $p' \mathrel{\underset{\sim}{\sqsubset}}_{RS} \mathcal{R} \mathrel{\underset{\sim}{\sqsubset}}_{RS} q'$ for some q'. Moreover, we also have $B_X\{q/X\} \equiv \langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'$, as desired.

Case 3 $B_X \equiv D_X \Box D'_X$.

If $B_X\{p/X\}$ is stable then we can proceed analogously to Case 2 with $\langle Y|E\rangle\{p/X\} \stackrel{\tau}{\not\longrightarrow}$. In the following, we consider the case $B_X\{p/X\} \stackrel{\tau}{\longrightarrow}$.

For the transitions $D_X\{p/X\}\Box D'_X\{p/X\} \equiv p_0 \stackrel{\tau}{\longrightarrow}_F \cdots \stackrel{\tau}{\longrightarrow}_F |p_n \equiv p'(n \geq 1)$, there exist two sequences of processes $t_0(\equiv D_X\{p/X\}), \dots, t_n$ and $s_0(\equiv D'_X\{p/X\}), \dots, s_n$ such that t_n, s_n are consistent and stable, $p_n \equiv t_n \Box s_n$, and for each $0 \leq i < n$, $p_i \equiv t_i \Box s_i$ and the last rule applied in $\mathcal{T}_i$ is

$$\text{either } \frac{t_i \stackrel{\tau}{\longrightarrow} t_{i+1}}{t_i \Box s_i \stackrel{\tau}{\longrightarrow} t_{i+1} \Box s_{i+1}} \text{ or } \frac{s_i \stackrel{\tau}{\longrightarrow} s_{i+1}}{t_i \Box s_i \stackrel{\tau}{\longrightarrow} t_{i+1} \Box s_{i+1}}.$$

For the former, $s_{i+1} \equiv s_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}'_i$ which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash t_i \stackrel{\tau}{\longrightarrow} t_{i+1}$. We use Ω_1 to denote the (finite) set of all these proof trees $\mathcal{T}'_i$. Similarly, for the latter, $t_{i+1} \equiv t_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}''_i$ which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash s_i \stackrel{\tau}{\longrightarrow} s_{i+1}$. We use Ω_2 to denote the (finite) set of all these proof trees $\mathcal{T}''_i$. It is obvious that Ω_1 is a proof forest for $D_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |t_n$, moreover,

$$Dep(\Omega_1) < Dep(\Omega).$$

Thus, by IH, we have $D_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'_1$ and $t_n \mathrel{\underset{\sim}{\sqsubset}}_{RS} \mathcal{R} \mathrel{\underset{\sim}{\sqsubset}}_{RS} q'_1$ for some q'_1. Similarly, for the transition $D'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |s_n$, we also have $D'_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'_2$ and $s_n \mathrel{\underset{\sim}{\sqsubset}}_{RS} \mathcal{R} \mathrel{\underset{\sim}{\sqsubset}}_{RS} q'_2$ for some q'_2. Then, by Theorem 4.3, it is easy to check that $p' \equiv t_n \Box s_n \mathrel{\underset{\sim}{\sqsubset}}_{RS} \mathcal{R} \mathrel{\underset{\sim}{\sqsubset}}_{RS} q'_1 \Box q'_2$. Moreover, we also have $B_X\{q/X\} \equiv D_X\{q/X\}\Box D'_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'_1 \Box q'_2$.

(**ALT-upto-2**) Suppose that $B_X\{p/X\}$ and $B_X\{q/X\}$ are stable. Let $B_X\{p/X\} \stackrel{a}{\Longrightarrow}_F |p'$ and Ω be its any proof forest. So, there exist $p_0, \dots, p_n (n \geq 1)$ such that

$$B_X\{p/X\} \equiv p_0 \stackrel{a}{\longrightarrow}_F p_1 \stackrel{\tau}{\longrightarrow}_F \cdots \stackrel{\tau}{\longrightarrow}_F |p_n \equiv p', \qquad (7.3.1)$$

and Ω exactly consists of proof trees $\mathcal{T}_i$ for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_i \stackrel{\alpha_i}{\longrightarrow} p_{i+1}$ with $i < n$, where $\alpha_0 = a$ and $\alpha_j = \tau(1 \leq j < n)$. We want to prove that there exists q' such that $B_X\{q/X\} \stackrel{a}{\Longrightarrow}_F |q'$ and $p' \mathrel{\underset{\sim}{\sqsubset}}_{RS} \mathcal{R} \mathrel{\underset{\sim}{\sqsubset}}_{RS} q'$ by induction on $Dep(\Omega)$. Since $B_X\{p/X\}$ is stable and X does not occur in any scope of conjunctions in B_X, the most top operator of B_X is neither disjunction nor conjunction. Thus, we distinguish five cases based on the form of B_X.

Case 1 $B_X \equiv X$.

Due to $B_X\{p/X\} \equiv p \stackrel{a}{\Longrightarrow}_F |p'$, we have $p \notin F$. Moreover, since $p(\equiv B_X\{p/X\})$ is

stable, we get $p \overset{\epsilon}{\Longrightarrow}_F |p$. Hence it follows from $p =_{RS} C_X\{p/X\}$ that

$$C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |s \text{ and } p \sqsubseteq_{RS} s \text{ for some } s.$$

Further, since X is strongly guarded and does not occur in any scope of conjunctions in C_X, by Lemma 5.17, there exists a stable context C'_X such that

(b.1) X is strongly guarded and does not occur in any scope of conjunctions in C'_X,
(b.2) $s \equiv C'_X\{p/X\}$, and
(b.3) $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow} C'_X\{q/X\}$.

Then it follows from $p \sqsubseteq_{RS} s \equiv C'_X\{p/X\}$ and $p \overset{a}{\Longrightarrow}_F |p'$ that

$$C'_X\{p/X\} \overset{a}{\Longrightarrow}_F |s' \text{ and } p' \sqsubseteq_{RS} s' \text{ for some } s'.$$

Since $p \overset{\tau}{\not\longrightarrow}$, by (b.1), Lemma 5.10 and 5.17, there exists a stable context C''_X such that

(c.1) $s' \equiv C''_X\{p/X\}$,
(c.2) X does not occur in any scope of conjunctions in C''_X, and
(c.3) $C'_X\{q/X\} \overset{a}{\longrightarrow}\overset{\epsilon}{\Longrightarrow} C''_X\{q/X\}$.

Moreover, since $q(\equiv B_X\{q/X\})$ is stable, so is $C''_X\{q/X\}$. Then, by (b.3) and (c.3), we have

$$C_X\{q/X\} \overset{\epsilon}{\Longrightarrow} |C'_X\{q/X\} \overset{a}{\Longrightarrow} |C''_X\{q/X\}.$$

Further, by Lemma 7.2 and 4.2, it follows from $p, q, C_X\{p/X\}, C'_X\{p/X\}, C''_X\{p/X\} \notin F$ that

$$C_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |C'_X\{q/X\} \overset{a}{\Longrightarrow}_F |C''_X\{q/X\}. \tag{7.3.2}$$

Then, since $C_X\{q/X\} =_{RS} q$ and $q \overset{\tau}{\not\longrightarrow}$, we get

$$C'_X\{q/X\} \sqsubseteq_{RS} q.$$

Further, due to (7.3.2), it follows that

$$B_X\{q/X\}(\equiv q) \overset{a}{\Longrightarrow}_F |q' \text{ and } C''_X\{q/X\} \sqsubseteq_{RS} q' \text{ for some } q'.$$

Moreover, $p' \sqsubseteq_{RS} s' \equiv C''_X\{p/X\}\mathcal{R}C''_X\{q/X\} \sqsubseteq_{RS} q'$, as desired.

Case 2 $B_X \equiv \alpha.D_X$.

So $\alpha = a$ and $D_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$. Clearly, $(D_X\{p/X\}, D_X\{q/X\}) \in R$. By (ALT-upto-1), there exists q' such that $D_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \sqsubseteq_{RS} \mathcal{R} \sqsubseteq_{RS} q'$. Moreover, it is evident that $\alpha.D_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'$.

Case 3 $B_X \equiv D_X \Box D'_X$.

W.l.o.g, assume that the last rule applied in $\mathcal{T}_0$ is $\frac{D_X\{p/X\} \overset{a}{\longrightarrow} p_1,\ D'_X\{p/X\} \overset{\tau}{\not\longrightarrow}}{D_X\{p/X\}\Box D'_X\{p/X\} \overset{a}{\longrightarrow} p_1}$. Then $\mathcal{T}_0$ has a proper subtree, say $\mathcal{T}'_0$, which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash D_X\{p/X\} \overset{a}{\longrightarrow} p_1$. Clearly, $\Omega' \triangleq \{\mathcal{T}'_0, \mathcal{T}_i : 1 \leq i \leq n-1\}$ is a proof forest for $D_X\{p/X\} \overset{a}{\Longrightarrow}_F |p'$ and $Dep(\Omega') < Dep(\Omega)$. Moreover, since $B_X\{q/X\}$ is stable, so are $D_X\{q/X\}$ and

$D'_X\{q/X\}$. Then, by IH, we have $D_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'$ for some q'. Moreover, $D'_X\{p/X\} \notin F$ because of $B_X\{p/X\} \notin F$, which, by Lemma 7.2, implies $D'_X\{q/X\} \notin F$. Hence $B_X\{q/X\} \equiv D_X\{q/X\}\Box D'_X\{q/X\} \notin F$, and $B_X\{q/X\} \equiv D_X\{q/X\}\Box D'_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'$, as desired.

Case 4 $B_X \equiv D_X \parallel_A D'_X$.

Then the last rule applied in $\mathcal{T}_0$ is one of the following three formats:

(1)$\frac{D_X\{p/X\} \overset{a}{\longrightarrow} t_1, D'_X\{p/X\} \overset{a}{\longrightarrow} s_1}{B_X\{p/X\} \parallel_A D_X\{p/X\} \overset{a}{\longrightarrow} t_1 \parallel_A s_1}$ with $a \in A$ and $p_1 \equiv t_1 \parallel_A s_1$;

(2)$\frac{D_X\{p/X\} \overset{a}{\longrightarrow} t_1,\ D'_X\{p/X\} \overset{\tau}{\not\longrightarrow}}{D_X\{p/X\} \parallel_A D'_X\{p/X\} \overset{a}{\longrightarrow} t_1 \parallel_A D'_X\{p/X\}}$ with $a \notin A$ and $p_1 \equiv t_1 \parallel_A D'_X\{p/X\}$;

(3)$\frac{D'_X\{p/X\} \overset{a}{\longrightarrow} s_1,\ D_X\{p/X\} \overset{\tau}{\not\longrightarrow}}{D_X\{p/X\} \parallel_A D'_X\{p/X\} \overset{a}{\longrightarrow} D_X\{p/X\} \parallel_A s_1}$ with $a \notin A$ and $p_1 \equiv D_X\{p/X\} \parallel_A s_1$.

We treat the first one, and the proof of the later two runs as one of Case 3. Clearly, $\mathcal{T}_0$ has two proper subtrees $\mathcal{T}'_0$ and $\mathcal{T}''_0$, which are proof trees for $D_X\{p/X\} \overset{a}{\longrightarrow} t_1$ and $D'_X\{p/X\} \overset{a}{\longrightarrow} s_1$ respectively. Moreover, for the transitions $p_1 \overset{\tau}{\longrightarrow} \cdots \overset{\tau}{\longrightarrow} |p_n$, there exist two processes sequences $t_1, \dots, t_n$ and $s_1, \dots, s_n$ such that t_n, s_n are stable, $p_n \equiv t_n \parallel_A s_n$, and for each $1 \le i < n$, $p_i \equiv t_i \parallel_A s_i$ and the last rule applied in $\mathcal{T}_i$ is

$$\text{either } \frac{t_i \overset{\tau}{\longrightarrow} t_{i+1}}{t_i \parallel_A s_i \overset{\tau}{\longrightarrow} t_{i+1} \parallel_A s_{i+1}} \text{ or } \frac{s_i \overset{\tau}{\longrightarrow} s_{i+1}}{t_i \parallel_A s_i \overset{\tau}{\longrightarrow} t_{i+1} \parallel_A s_{i+1}}.$$

For the former, $s_{i+1} \equiv s_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}'_i$ which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash t_i \overset{\tau}{\longrightarrow} t_{i+1}$. We use Ω_1 to denote the (finite) set of all these proof tree $\mathcal{T}'_i$. Similarly, for the latter, $t_{i+1} \equiv t_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}''_i$ which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash s_i \overset{\tau}{\longrightarrow} s_{i+1}$. We use Ω_2 to denote the (finite) set of all these proof tree $\mathcal{T}''_i$. Clearly, $\Omega' \triangleq \{\mathcal{T}'_0\} \cup \Omega_1$ is a proof forest for $D_X\{p/X\} \overset{a}{\Longrightarrow}_F |t_n$ and $Dep(\Omega') < Dep(\Omega)$. Thus, by IH, we have $D_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'_1$ and $t_n \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'_1$ for some q'_1. Similarly, for the transition $D'_X\{p/X\} \overset{a}{\Longrightarrow}_F |s_n$, we also have $D'_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'_2$ and $s_n \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'_2$ for some q'_2. Therefore, by Theorem 4.3, we obtain $p' \equiv t_n \parallel_A s_n \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'_1 \parallel_A q'_2$. Moreover, it is not difficult to see that $B_X\{q/X\} \equiv D_X\{q/X\} \parallel_A D'_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'_1 \parallel_A q'_2$ because of $B_X\{q/X\} \overset{\tau}{\not\longrightarrow}$, $D_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'_1$ and $D'_X\{q/X\} \overset{a}{\Longrightarrow}_F |q'_2$.

Case 5 $B_X \equiv \langle Y|E\rangle$.

Clearly, the last rule applied in $\mathcal{T}_0$ is $\frac{\langle t_Y|E\rangle\{p/X\} \overset{a}{\longrightarrow} p_1}{\langle Y|E\rangle\{p/X\} \overset{a}{\longrightarrow} p_1}$. Hence $\mathcal{T}_0$ contains a proper subtree, say $\mathcal{T}'_0$, which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash \langle t_Y|E\rangle\{p/X\} \overset{a}{\longrightarrow} p_1$, and $Dep(\mathcal{T}'_0) < Dep(\mathcal{T}_0)$. So, $\Omega' \triangleq \{\mathcal{T}'_0, \mathcal{T}_i : 1 \le i < n\}$ is a proof forest for $\langle t_Y|E\rangle\{p/X\} \overset{a}{\Longrightarrow}_F |p'$ and $Dep(\Omega') < Dep(\Omega)$. Then, by IH, we have $\langle t_Y|E\rangle\{q/X\} \overset{a}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'$ for some q', moreover, $B_X\{q/X\} \equiv \langle Y|E\rangle\{q/X\} \overset{a}{\Longrightarrow}_F |q'$, as desired.

(**ALT-upto-3**) Let $B_X\{p/X\}$ and $B_X\{q/X\}$ be stable and $B_X\{p/X\} \notin F$. We shall prove $\mathcal{I}(B_X\{p/X\}) \supseteq \mathcal{I}(B_X\{q/X\})$, the converse inclusion may be proved in a similar

manner and omitted. Assume that $B_X\{q/X\} \xrightarrow{a} q'$. Then, for such a-labelled transition, by Lemma 5.9, there exist B'_X, $B'_{X,\widetilde{Y}}$ and $B''_{X,\widetilde{Y}}$ with $X \notin \widetilde{Y}$ that satisfy (CP-a-1) – (CP-a-4). For the case $\widetilde{Y} = \emptyset$, it immediately follows from (CP-a-3-iii) that $B_X\{p/X\} \xrightarrow{a} B''_{X,\widetilde{Y}}\{p/X\}$.

Next we handle the case $\widetilde{Y} \neq \emptyset$. In this situation, by (CP-a-3-iii), to complete the proof, it suffices to prove that $\mathcal{I}(p) = \mathcal{I}(q)$. By (CP-$a$-1) and (CP-$a$-3-i), we have

$$B_X\{r/X\} \Rrightarrow B'_{X,\widetilde{Y}}\{r/X, r/\widetilde{Y}\} \text{ for any } r.$$

Then, since $B_X\{p/X\}$ and $B_X\{q/X\}$ are stable, by $\widetilde{Y} \neq \emptyset$, (CP-$a$-2) and Lemma 5.12 and 5.4, it follows that both p and q are stable. Hence $p \stackrel{\epsilon}{\Longrightarrow}_F |p$ by $p \notin F$. Then, due to $p =_{RS} C_X\{p/X\}$, we have

$$C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ and } p \sqsubset\!\!\!\!\sim_{RS} s \text{ for some } s.$$

For the transition above, since X is strongly guarded in C_X, by Lemma 5.17, there exists a stable context D_X such that

(d.1) $s \equiv D_X\{p/X\} \stackrel{\tau}{\not\longrightarrow}$,

(d.2) X is strongly guarded and does not occur in any scope of conjunctions in D_X, and

(d.3) $C_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow} D_X\{q/X\}$.

Hence $\mathcal{I}(p) = \mathcal{I}(D_X\{p/X\})$ by (d.1), $p \sqsubset\!\!\!\!\sim_{RS} s$ and $p \notin F$. Moreover, by (d.1), (d.2) and Lemma 5.10, we have $D_X\{q/X\} \stackrel{\tau}{\not\longrightarrow}$ and

$$\mathcal{I}(p) = \mathcal{I}(D_X\{p/X\}) = \mathcal{I}(D_X\{q/X\}).$$

On the other hand, since $s \equiv D_X\{p/X\} \notin F$, by Lemma 7.2 and $p, q \notin F$, we obtain $D_X\{q/X\} \notin F$. So, $C_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |D_X\{q/X\}$. Further, it follows from $q =_{RS} C_X\{q/X\}$ and $q \stackrel{\tau}{\not\longrightarrow}$ that $D_X\{q/X\} \sqsubset\!\!\!\!\sim_{RS} q$. Hence $\mathcal{I}(D_X\{q/X\}) = \mathcal{I}(q)$ because of $D_X\{q/X\} \notin F$. Therefore, $\mathcal{I}(p) = \mathcal{I}(D_X\{p/X\}) = \mathcal{I}(D_X\{q/X\}) = \mathcal{I}(q)$, as desired. □

The next lemma is the crucial step in the demonstrating the assertion that $\langle X|X = t_X\rangle$ is a consistent solution for a given equation $X =_{RS} t_X$ whenever consistent solutions exist.

Lemma 7.4. For any term t_X where X is strongly guarded and does not occur in any scope of conjunctions, if $q =_{RS} t_X\{q/X\}$ for some $q \notin F$ then $\langle X|X = t_X\rangle \notin F$.

Proof. Assume $p =_{RS} t_X\{p/X\}$ for some $p \notin F$. Then $t_X\{p/X\} \notin F$. Set

$$\Omega = \left\{ B_Y\{\langle X|X = t_X\rangle/Y\} : \begin{array}{l} B_Y\{p/Y\} \notin F \text{ and } Y \text{ does not occur in any scope of} \\ \text{conjunctions in } B_Y \end{array} \right\}.$$

It is obvious that $\langle X|X = t_X\rangle \in \Omega$ by taking $B_Y \triangleq Y$. Thus we intend to show that $\Omega \cap F = \emptyset$. Assume $C_Y\{\langle X|X = t_X\rangle/Y\} \in \Omega$. Let $\mathcal{T}$ be any proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_Y\{\langle X|X = t_X\rangle/Y\}F$. Similar to Lemma 6.3, it is enough to prove that $\mathcal{T}$ has a proper subtree with the root labelled with sF for some $s \in \Omega$, which is a routine case analysis based on the last rule applied in $\mathcal{T}$. Here we treat only

two cases as samples.

Case 1 $C_Y \equiv Y$.

Then $C_Y\{\langle X|X=t_X\rangle/Y\} \equiv \langle X|X=t_X\rangle$. Clearly, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_X|X=t_X\rangle F}{\langle X|X=t_X\rangle F} \text{ or } \frac{\{rF : \langle X|X=t_X\rangle \overset{\epsilon}{\Longrightarrow} |r\}}{\langle X|X=t_X\rangle F}.$$

For the former, $\mathcal{T}$ has a proper subtree with the root labelled with $\langle t_X|X=t_X\rangle F$, moreover, $\langle t_X|X=t_X\rangle \equiv t_X\{\langle X|X=t_X\rangle/X\} \in \Omega$ due to $t_X\{p/X\} \notin F$, as desired.

For the latter, if $\langle X|X=t_X\rangle \overset{\tau}{\not\longrightarrow}$, then, in $\mathcal{T}$, the unique node directly above the root is labelled with $\langle X|X=t_X\rangle F$, moreover $\langle X|X=t_X\rangle \in \Omega$, as desired. In the following, we consider the nontrivial case $\langle X|X=t_X\rangle \overset{\tau}{\longrightarrow}$. Since $t_X\{p/X\} \notin F$, by Theorem 4.2, we get $t_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p'. For this transition, since X is strongly guarded and does not occur in any scope of conjunctions, by Lemma 5.17, there exists a stable context B_X such that

(a.1) X is strongly guarded and does not occur in any scope of conjunctions,
(a.2) $p' \equiv B_X\{p/X\}$, and
(a.3) $t_X\{\langle X|X=t_X\rangle/X\} \overset{\epsilon}{\Longrightarrow} B_X\{\langle X|X=t_X\rangle/X\}$.

Since $p' \equiv B_X\{p/X\} \overset{\tau}{\not\longrightarrow}$, by (a.1) and Lemma 5.10, $B_X\{\langle X|X=t_X\rangle/X\} \overset{\tau}{\not\longrightarrow}$. Then it follows from (a.3) and $\langle X|X=t_X\rangle \overset{\tau}{\longrightarrow}$ that $\langle X|X=t_X\rangle \overset{\epsilon}{\Longrightarrow} |B_X\{\langle X|X=t_X\rangle/X\}$. Hence $\mathcal{T}$ has a proper subtree with the root labelled with $B_X\{\langle X|X=t_X\rangle/X\}F$, moreover, $B_X\{\langle X|X=t_X\rangle/X\} \in \Omega$ because of $p' \notin F$, (a.1) and (a.2).

Case 2 $C_Y \equiv \langle Z|E\rangle$.

So $C_Y\{\langle X|X=t_X\rangle/Y\} \equiv \langle Z|E\{\langle X|X=t_X\rangle/Y\}\rangle$. Then the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Z|E\rangle\{\langle X|X=t_X\rangle/Y\}F}{\langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\}F}(Z=t_Z\in E) \text{ or } \frac{\{rF : \langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\} \overset{\epsilon}{\Longrightarrow} |r\}}{\langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\}F}.$$

For the first alternative, by Lemma 4.1(8), it follows from $\langle Z|E\rangle\{p/Y\} \notin F$ that $\langle t_Z|E\rangle\{p/Y\} \notin F$. Since Y does not occur in any scope of conjunctions in $\langle Z|E\rangle$, by Lemma 5.2(5), nor does it in $\langle t_Z|E\rangle$. Therefore $\langle t_Z|E\rangle\{\langle X|X=t_X\rangle/Y\} \in \Omega$, as desired.

For the second alternative, since $\langle Z|E\rangle\{p/Y\} \notin F$ and $p =_{RS} t_X\{p/X\}$, we get $\langle Z|E\rangle\{t_X\{p/X\}/Y\} \notin F$ by Theorem 6.1. So $\langle Z|E\rangle\{t_X\{p/X\}/Y\} \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p'. Then, for this transition, by Lemma 5.17, there exist processes q_W with $W \in \widetilde{W}$ and a context $D_{Y,\widetilde{W}}$ with $Y \notin \widetilde{W}$ such that

(b.1) $t_X\{p/X\} \overset{\tau}{\Longrightarrow} |q_W$ with $W \in \widetilde{W}$ and $p' \equiv D_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{q_W}/\widetilde{W}\}$,
(b.2) Y and each $W(\in \widetilde{W})$ are strongly guarded and do not occur in any scope of conjunctions in $D_{Y,\widetilde{W}}$, and
(b.3) $\langle Z|E\rangle\{r/Y\} \overset{\epsilon}{\Longrightarrow} D_{Y,\widetilde{W}}\{r/Y, \widetilde{r_W}/\widetilde{W}\}$ for any r and r_W with $W \in \widetilde{W}$ such that $r \overset{\tau}{\Longrightarrow} r_W$.

Then, since X is strongly guarded and does not occur in any scope of conjunctions in

t_X, by Lemma 5.17 and 5.10, for each transition $t_X\{p/X\} \stackrel{\tau}{\Longrightarrow} |q_W$, there exists a stable context t_X^W such that

(c.1) X is strongly guarded and does not occur in any scope of conjunctions in t_X^W,
(c.2) $q_W \equiv t_X^W\{p/X\}$, and
(c.3) $t_X\{\langle X|X=t_X\rangle/X\} \stackrel{\tau}{\Longrightarrow} |t_X^W\{\langle X|X=t_X\rangle/X\}$.

For the simplicity of notation, we let Q_W stand for $t_X^W\{\langle X|X=t_X\rangle/X\}$ with $W \in \widetilde{W}$. So, by (c.3), $\langle X|X=t_X\rangle \stackrel{\tau}{\Longrightarrow} |Q_W$ for each $W \in \widetilde{W}$. Hence it follows from (b.3) that

$$\langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\} \stackrel{\epsilon}{\Longrightarrow} D_{Y,\widetilde{W}}\{\langle X|X=t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\}. \tag{7.4.1}$$

By (b.2) and (c.1), it is not difficult to see that X is strongly guarded and does not occur in any scope of conjunctions in $D_{Y,\widetilde{W}}\{t_X/Y, \widetilde{t_X^W}/\widetilde{W}\}$. So, by Lemma 5.10 and $p' \equiv D_{Y,\widetilde{W}}\{t_X/Y, \widetilde{t_X^W}/\widetilde{W}\}\{p/X\} \stackrel{\tau}{\not\longrightarrow}$, we get

$$D_{Y,\widetilde{W}}\{t_X/Y, \widetilde{t_X^W}/\widetilde{W}\}\{\langle X|X=t_X\rangle/X\} \stackrel{\tau}{\not\longrightarrow}.$$

Hence $D_{Y,\widetilde{W}}\{\langle X|X=t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\} \stackrel{\tau}{\not\longrightarrow}$ by Lemma 5.6 and $\mathcal{I}(\langle X|X=t_X\rangle) = \mathcal{I}(t_X\{\langle X|X=t_X\rangle/X\})$. Then $\mathcal{T}$ has a proper subtree with the root labelled with $D_{Y,\widetilde{W}}\{\langle X|X=t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\}F$ due to (7.4.1). On the other hand, by Theorem 6.1 and $p =_{RS} t_X\{p/X\}$, it follows from $p' \equiv D_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{t_X^W\{p/X\}}/\widetilde{W}\} \notin F$ that $D_{Y,\widetilde{W}}\{p/Y, \widetilde{t_X^W\{p/X\}}/\widetilde{W}\} \notin F$. Set

$$D'_Y \triangleq D_{Y,\widetilde{W}}\{\widetilde{t_X^W\{Y/X\}}/\widetilde{W}\}.$$

Therefore, $D_{Y,\widetilde{W}}\{\langle X|X=t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\} \equiv D'_Y\{\langle X|X=t_X\rangle/Y\} \in \Omega$, as desired. □

We now have the below assertion which states that given an equation $X =_{RS} t_X$ satisfying some conditions, $\langle X|X=t_X\rangle$ is the unique consistent solution for it whenever consistent solutions exist.

Theorem 7.1 (Unique solution). For any $p, q \notin F$ and t_X where X is strongly guarded and does not occur in any scope of conjunctions, if $p =_{RS} t_X\{p/X\}$ and $q =_{RS} t_X\{q/X\}$ then $p =_{RS} q$. Moreover, $\langle X|X=t_X\rangle$ is the unique consistent solution modulo $=_{RS}$ for the equation $X =_{RS} t_X$ whenever consistent solutions exist.

Proof. If $p \bowtie q$ then $p =_{RS} q$ follows from Lemma 7.3, otherwise, w.l.o.g, we assume that p is stable and q is not. By Theorem 6.1, $\tau.p =_{RS} p =_{RS} t_X\{p/X\} =_{RS} t_X\{\tau.p/X\}$. Then, by Lemma 7.3, it follows from $\tau.p, q \notin F$, $\tau.p \bowtie q$, $\tau.p =_{RS} t_X\{\tau.p/X\}$ and $q =_{RS} t_X\{q/X\}$ that $\tau.p =_{RS} q$. Hence $p =_{RS} q$.

Suppose that $X =_{RS} t_X$ has consistent solutions. It is obvious that $\langle X|X=t_X\rangle =_{RS} t_X\{\langle X|X=t_X\rangle/X\}$ due to $\langle X|X=t_X\rangle \Rrightarrow_1 \langle t_X|X=t_X\rangle \equiv t_X\{\langle X|X=t_X\rangle/X\}$ and Lemma 5.16. Further, by Lemma 7.4, $\langle X|X=t_X\rangle$ is the unique consistent solution of the equation $X =_{RS} t_X$. □

As an immediate consequence, we have

Corollary 7.1. For any term t_X where X is strongly guarded and does not occur in any scope of conjunctions, then the equation $X =_{RS} t_X$ has consistent solutions iff $\langle X|X = t_X\rangle \notin F$.

Proof. Immediately by Theorem 7.1. □

We conclude this section with providing a brief discussion. For Theorem 7.1, the condition that X is strongly guarded can not be relaxed to that X is weakly guarded. For instance, consider the equation $X =_{RS} \tau.X$, it has infinitely many consistent solutions. In fact, for any p, it always holds that $p =_{RS} \tau.p$. Moreover, the condition that $p, q \notin F$ is also necessary. For example, both $\langle X|X = a.X\rangle$ and $\perp$ are solutions for the equation $X =_{RS} a.X$, but they are not equivalent modulo $=_{RS}$. Finally, at present, we do not know whether this result still holds if we omit the hypothesis that X does not occur in any scope of conjunctions in t_X.

8. Conclusions and future work

This paper considers recursive operations over LLTSs in pure process-algebraic style. We propose the process calculus CLL_R, which is obtained from CLL by adding recursive operations. Although CLL contains logic operators $\wedge$ and $\vee$ over processes, due to lack of modal operators, it does not afford describing abstract properties of concurrent systems. However, under the mild condition that the set of actions is finite, we can integrate standard temporal operators *always* and *unless* into process calculuses containing $\wedge$ and $\vee$ in a recursive manner (Zhu *et al.* 2013). In detail, the processes

$$\langle X|X = q \vee (p \wedge (\bigwedge_{a\in Act} \lceil a\rceil X))\rangle$$

and

$$\langle X|X = p \wedge (\bigwedge_{a\in Act} \lceil a\rceil X)\rangle$$

capture temporal operators *unless* and *always* respectively, where $\lceil a\rceil X$ is the abbreviation of

$$(\bigvee_{a\in A\subseteq Act}((\Box_{b\in A-\{a\}} b.true)\Box a.X)) \vee (\bigvee_{a\notin A\subseteq Act}(\Box_{b\in A} b.true))$$

and *true* is the abbreviation of

$$\langle X|X = \bigvee_{A\subseteq Act}\Box_{a\in A} a.X\rangle.$$

For a fuller treatment of this issue we refer the reader to (Zhu *et al.* 2013). Therefore we can make a conclusion that, under the assumption that *Act* is finite, the calculus CLL_R is enough to express safety properties considered in (Lüttgen and Vogler 2011).

This paper have showed that the behavior relation $\sqsubseteq_{RS}$ is precongruent w.r.t all operations in CLL_R, which reveals that this calculus supports compositional reasoning. Moreover, we also provide a theorem on the uniqueness of consistent solution for a given

equation $X =_{RS} t_X$ where X is required to be strongly guarded and does not occur in any scope of conjunctions in t_X.

One of future work is to find a (ground) complete proof system for regular processes in CLL_R along lines adopted in (Milner 1989). Here a process is said to be regular if its LTS has only finitely many states and transitions. To this end, it is necessary to adopt the restriction that recursive variables do not occur in any scope of conjunctions in recursive specifications. Otherwise, non-regular expressions would occur, for instance, consider the process $\langle X|X = a.X \wedge \tau.a.X\rangle$. Thus we think that the result on the uniqueness of consistent solution obtained in this paper may be enough for this aim.

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